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Asymptotic Convergence of Optimal Policies for Resource Management with Application to Harvesting of Multiple Species Forest

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We study the long-term behavior of the optimal harvesting policies for a mixed forest composed by multiple species of different maturity ages. This model is a prototype for the exploitation of a finite resource such as land or space, which can be allocated to different activities that produce their revenue after certain delays at which the resource is liberated for reuse. We prove the existence and uniqueness of a *sustainable state*, and we discuss the conditions under which an optimal trajectory converges in the long run toward this state or toward the set of optimal periodic cycles. We also analyze different situations in which the convergence occurs in finite time.

Key words: forest management; intertemporal resource allocation; optimal control; discrete time; infinite horizon; turnpike property; Lyapunov stability
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In this paper, we investigate a model for the optimal management of a finite resource, which can be allocated to different activities that, after a fixed delay, produce their revenue and liberate the resource for immediate reuse. For the sake of concreteness, we state the model in the context of a forest management problem that seeks the optimal sowing and harvesting policy for a given land surface that can be allocated among different forest species. An alternative context in which the model might apply is the management of space in a storage warehouse where customers may rent a fraction of the total space under multiple hiring contracts with different delays. A third example coming from finance concerns the management of the so-called *"interest in possession trusts,"* where capital is invested on different financial instruments with rents perceived after fixed delays at which the beneficiary receives all the income but without any rights to the capital, which must be reinvested keeping the total capital fixed. Our model concerns a perfectly divisible resource so it does not apply to the management of indivisible units such as winery barrels or fish farm tanks that cannot be split into arbitrary fractions.

In 1849, Faustmann [2] considered the problem of finding the economic value of a forest. Using periodic harvesting policies and discounted utilities, he gave an expression for the net present value of an even-aged forest stand and raised the question of finding an optimal policy that would take into account the growth of the trees. The question was solved by Ohlin in 1921 (Ohlin [10]) characterizing the optimal rotation period, which came to be known as the *Faustmann rotation age*. The extension to a forest with many even-aged stands was already considered at that time, but its complete resolution remains open even today. Faustmann's [2] ideas were extremely influential and inspired various harvesting rules with a long-run behavior that guarantees a sustainable and regular flow of timber. In particular, optimal harvesting policies have been studied numerically with different types of even-flow constraints, or requiring convergence to a steady state such as the *normal forest* or *sustainable state* in which the land is evenly allocated among all the age classes with a Faustmann [2] rotation period.

More recently, Mitra and Wan [8] reconsidered the optimal harvesting of a multiaged single species forest in a form well suited for the application of the general theory of intertemporal allocation. Their results partially contradict the steady-state paradigm. Indeed, although their main result proves the existence of a sustainable state, which is invariant under an optimal policy, they also found examples where the optimal solution is a periodic cycle and does not converge to the sustainable state. The issue was further investigated by Salo and Tahvonen [14] showing that every initial condition close enough to the sustainable state yields a periodic optimal trajectory, so that this state is not even a local attractor. They conjectured that the long-run behavior of any optimal policy would be periodic, proving this result for the case of a two-stand forest. The same authors extended the model to include the possibility of allocating land to an annual alternative use, showing that the optimal periodic cycles disappear when it is optimal to allocate part of the land to the annual use, in which case the sustainable state turns out to be a local attractor (cf. Salo and Tahvonen [13, 15]). The optimal management of a one species forest was also studied by Rapaport et al. [12] using a model where harvest is restricted a priori to mature trees older than a certain age, and the growth after maturity is neglected. They defined a *greedy policy* as one where each tree is harvested as soon as it reaches maturity, showing that every optimal trajectory becomes greedy and periodic in finite time.

In this paper, we consider a mixed forest composed by several species of different maturity ages, with harvest restricted to mature trees as in the Rapaport et al. [12] model. This work was part of the doctoral dissertation (Piazza [11]) where several alternative forest management models were investigated. Our main goal is to study the *global* asymptotic behavior of the optimal policies, for which we use a Lyapunov-like methodology. In Theorem 3.3, we prove that an optimally managed forest converges toward the set of states for which the corresponding optimal trajectories are periodic. Under a mild additional condition on the maturity ages, this implies the convergence of the forest toward the sustainable state. Additionally, in Propositions 3.1 and 3.2, we investigate the transient phase by finding sufficient conditions under which any optimal trajectory becomes greedy in finite time, which provides some insight on what to do today and not only in the asymptotic regime.

Asymptotic convergence of optimal trajectories and turnpike results have been established for wide classes of dynamic optimization problems, most of them issued from the literature on economic growth models. For a survey of the general results available, we refer to Le Van et al. [3], McKenzie [5, 6], Zaslavski [17], and references therein. A distinguishing feature in our context is that we obtain global convergence in a discounted utility framework with *no restriction on the discount factor*, while most long-run characterizations are either local or assume discount factors close to one. This is a notable fact because dynamic optimization models under strong discounting often exhibit a complex behavior, including chaos (see, for example, Boldrin and Montrucchio [1], Le Van et al. [3], Majumdar et al. [4], Mitra and Nishimura [7], Montrucchio [9]). This more regular behavior comes from the fact that the forest evolution has a natural periodic structure determined by the least common multiple N of the maturity ages of the species. This allows to construct a Lyapunov function that does not increase at every time step but every N periods, which suffices nevertheless to perform the asymptotic study.

The paper is structured as follows. Section 1 introduces the optimization model to be solved, and then §2 describes some periodic optimal trajectories, including a precise definition of the sustainable state and the greedy periodic cycles (GPCs). In §3, we state our main results on the convergence of an optimally managed forest toward the sustainable state when this state allocates the land to species whose maturity ages are co-prime, and convergence toward the set of GPCs otherwise. Finally, in §4, we discuss the finite-time convergence for a two-species forest.

1. Model formulation. Let us consider a forest of total area S occupied by k different species $I=\{1,\ldots,k\}$ with maturity ages of $n_1,\ldots,n_k$ years, respectively. In contrast with the case of wild forests, the state of a forest plantation may be described by specifying the areas occupied by trees of different ages and species. For each period $t\in\mathbb{N}$, we denote $x_t^i\ge 0$ the area of species $i\in I$ that reaches its maturity in year t, and $\bar{x}_t^i\ge 0$ the area occupied by overmature trees (older than n_i). Using a single state variable per species to represent the overmature trees conveys the underlying assumption that the growth of trees is negligible beyond maturity. In each period, we must decide how much land $u_t^i\ge 0$ to harvest and how to reallocate this land to new seedlings. Assuming that only mature trees can be harvested, we must have $u_t^i\le \bar{x}_t^i+x_t^i$, and then the area not harvested in that period will comprise the overmature trees at the next step; namely,

$$\bar{x}_{t+1}^i=\bar{x}_t^i+x_t^i-u_t^i. \tag{1}$$

The total harvested area $\sum_{i\in I}u_t^i$ is allocated to new seedlings that will reach maturity in years $t+n_i$, respectively, which is expressed by the equation

$$\sum_{i\in I}x_{t+n_i}^i=\sum_{i\in I}u_t^i. \tag{2}$$

The total benefit obtained from the harvests is given by the value

$$V=\sum_{i\in I}\sum_{t=0}^{\infty}b^t U_i(u_t^i), \tag{3}$$

where $b\in(0,1)$ is a discount rate and the utility functions $U_i\colon\mathbb{R}\to\mathbb{R}$ are smooth, increasing, and strictly concave for each $i\in I$. The problem is then to find a sequence of harvests $u_t^i\ge 0$, which maximizes the value (3), while keeping the state variables x_t^i and $\bar{x}_t^i$ nonnegative subject to the constrains (1) and (2).

Remark 1.1. Our model arbitrarily restricts the harvests to mature trees of ages n_i and over. The n_is are themselves fixed exogenously and do not come out from an optimization exercise that would take into account the interplay between the growth of trees and the discount factor (see Remark 2.1 for a possible methodology to choose the maturity ages n_i). This is an important difference with respect to the model studied by Mitra and Wan [8], which considers the biomass growth and allows harvesting at any age. In our setting, we only consider the timber content at age n_i, which allows to describe the model using *area* as the state variable. Consequently, the utility functions U_i must account simultaneously for the market price and the timber content per unit area at the age of maturity.

Let us denote $\mathbf{x}^i$, $\bar{\mathbf{x}}^i$, $\mathbf{u}^i$ the sequences of states and controls. Since all the areas are nonnegative and smaller than S, these sequences belong to the l^∞ ball of radius S and centered at the origin B_S^∞. An alternative representation of the forest in terms of the age distribution at time t is provided by the *state* $\mathbb{X}_t = (X_t^1, \ldots, X_t^k)$, where $X_t^i = (x_{t+n_i-1}^i, x_{t+n_i-2}^i, \ldots, x_t^i, \bar{x}_t^i)$ describes the areas occupied in year t by trees of species i with ages $1, 2, \ldots, n_i$ and over n_i. Using this state variable, the evolution consists of an age shift dynamics except for the first and last components of each vector X_t^i, which represent the sowing at period t and the unharvested area, respectively. Although we will not use these dynamics explicitly, the state $\mathbb{X}_t$ will be useful in describing the asymptotic behavior of the forest. Notice that we do not control $\mathbb{X}_0$, which corresponds to the initial state reflecting the age-class composition of the forest at time $t = 0$. Hence the problem to be solved may be stated as

$$P(\mathbb{X}_0)\begin{cases} \text{maximize} \quad V = \sum_{i\in I}\sum_{t=0}^{\infty} b^t U_i(u_t^i), \\ \text{subject to} \quad (1) \text{ and } (2) \\ \qquad\qquad \mathbf{x}^i, \bar{\mathbf{x}}^i, \mathbf{u}^i \in l_+^\infty \text{ with } \mathbb{X}_0 \text{ given.} \end{cases}$$

We denote Δ the set of all initial states $\mathbb{X}_0$ such that $\sum_{i\in I}[\bar{x}_0^i + \sum_{t=0}^{n_i-1} x_t^i] = S$. Clearly, the area balance constraints imply that $\mathbb{X}_t \in \Delta$ for all $t \in \mathbb{N}$. We also denote Δ^0 the set of states with $\bar{x}_0^i = 0$ for all $i \in I$, and we observe that an initial state $\mathbb{X}_0 \in \Delta$ yields the same optimal value and harvesting policy as $\widetilde{\mathbb{X}}_0 \in \Delta^0$, where $\widetilde{X}_0^i = (x_{n_i-1}^i, \ldots, x_1^i, x_0^i + \bar{x}_0^i, 0)$.

Intuitively, the presence of a discount factor suggests an advantage for control strategies that harvest all the mature trees as soon as possible; that is to say, an optimal solution should lead to $\mathbb{X}_t \in \Delta^0$. However, this is balanced by the concavity of U_i, which favors homogeneous harvests at each stage. Hence, keeping some trees beyond maturity might be convenient to transfer area between different age classes so as to reshape the forest into a more homogeneous state. The difficulty in solving $P(\mathbb{X}_0)$ comes precisely from the trade-offs among these two conflicting forces. A natural conjecture is that the area transfers should occur only during an initial phase after which the optimal policy should drive the forest to an homogeneous state. While this is not always the case, in the next sections, we investigate the asymptotic behavior of the state $\mathbb{X}_t$ for an optimally managed forest. Before proceeding, we briefly discuss the existence and uniqueness of optimal policies. The arguments are rather standard so the reader may safely skip the proofs.

Proposition 1.1. *For each $\mathbb{X}_0 \in \Delta$, the problem $P(\mathbb{X}_0)$ has optimal solutions.*

Proof. The feasible set of $P(\mathbb{X}_0)$ is nonempty as it follows by considering the periodic trajectory that results from a control strategy in which all mature areas are harvested and then sowed with the same species as before; that is to say, $u_t^i = \bar{x}_t^i + x_t^i = x_{t+n_i}^i$. On the other hand, we already observed that $\mathbf{x}^i, \bar{\mathbf{x}}^i, \mathbf{u}^i \in B_S^\infty$ and since this ball is $\sigma(l^\infty, l^1)$-compact while the linear constraints in $P(\mathbb{X}_0)$ define a closed convex subset of $(B_S^\infty)^{3k}$, we deduce that the feasible set is weak* compact. Hence, it suffices to prove that the objective function is weak* upper semicontinuous. The functions $u^i \mapsto \sum_{t=0}^\infty b^t U_i(u_t^i)$ are concave and strongly continuous from l^∞ to $\mathbb{R}$, and therefore they are weakly u.s.c. However, we must consider the weak* topology so we proceed directly. Consider a weak* convergent net $\mathbf{u}^\alpha \overset{*}{\rightharpoonup} \mathbf{u}^i$, so that $u_t^\alpha \to u_t^i$ for all $t \in \mathbb{N}$, and take $\beta, \delta \in \mathbb{R}$ with $U_i(z) \le \beta + \delta z$. For any fixed $N \in \mathbb{N}$, we may write

$$\begin{aligned} \limsup_\alpha \sum_{t=0}^\infty b^t U_i(u_t^\alpha) &= \sum_{t=0}^N b^t U_i(u_t^i) + \limsup_\alpha \sum_{t=N+1}^\infty b^t U_i(u_t^\alpha) \\ &\le \sum_{t=0}^N b^t U_i(u_t^i) + \limsup_\alpha \sum_{t=N+1}^\infty b^t [\beta + \delta u_t^\alpha] \\ &= \sum_{t=0}^N b^t U_i(u_t^i) + \sum_{t=N+1}^\infty b^t [\beta + \delta u_t^i] \end{aligned}$$

the last equality since $(b^t\delta)_{t\in\mathbb{N}} \in l^1(\mathbb{N})$ and $\mathbf{u}^\alpha \xrightarrow{*} \mathbf{u}^i$. Letting $N \to \infty$, we conclude

$$\limsup_{\alpha} \sum_{t=0}^{\infty} b^t U_i(u_t^\alpha) \leq \sum_{t=0}^{\infty} b^t U_i(u_t^i). \quad \square$$

The strict concavity of the U_i's implies that the harvests $\mathbf{u}^i$ are uniquely determined, though uniqueness of $\mathbf{x}^i$ and $\bar{\mathbf{x}}^i$ does not follow directly from this. However, if we restrict to greedy trajectories in which $\bar{\mathbf{x}}^i = 0$, then $\mathbf{x}^i$ is uniquely determined. The latter remains true even if one utility function; say U_{i_0}, is merely concave instead of strictly concave. Indeed, using (2), we may eliminate the variables $u_t^{i_0}$ and write a reduced optimization problem, which gives the uniqueness of the harvests $\{\mathbf{u}^i: i \neq i_0\}$. If we then consider greedy strategies, all the area flows $\mathbf{x}^i$ and $\mathbf{u}^i$ (including $i = i_0$) are uniquely determined.

The existence of multipliers in l^1 (i.e., a solution for a dual problem) is a delicate issue since we lack a constraint qualification. However, in the next sections, we consider situations where the optimal strategy is periodic and for which one can explicitly find multipliers. It is worth mentioning that the results on optimal economic growth given by McKenzie [5, 6], as well as the general duality theorem of Weitzman [16], do not seem to apply in our setting. Indeed, the *free disposal hypothesis* in those models requires that if one can go from state $\mathbb{X}$ to $\mathbb{X}'$ in one time step, then it is also possible to go from any state "$\mathbb{Y} \geq \mathbb{X}$" to $\mathbb{X}'$ obtaining at least the same benefit. However, because of the area balance constraints, the natural state space for our dynamics is the set Δ, which is not endowed with a natural order.

2. Stationary optimal trajectories. Some initial states $\mathbb{X}_0$ lead to optimal trajectories, which are *periodic* or even *invariant*. Our goal in later sections is precisely to understand the extent to which an optimally managed forest starting from an arbitrary initial condition may or may not converge to such optimally stationary states.

2.1. Sustainable state. We begin by introducing the notion of a sustainable state, which corresponds intuitively to a forest with an age distribution at which it is optimal to stay forever.

Definition 2.1. A state $\mathbb{X}^* \in \Delta$ is called *sustainable* if it is invariant under an optimal harvesting-sowing policy; that is to say, the maximum benefit in problem $P(\mathbb{X}_0)$ when the initial state is given by $\mathbb{X}_0 = \mathbb{X}^*$, is obtained with a harvesting-sowing policy that leaves the state invariant: $\mathbb{X}_t = \mathbb{X}^*$ for all $t \in \mathbb{N}$.

The existence of a sustainable state is not completely obvious. Clearly, such a state must be of the form $X^i = (x^i, \ldots, x^i, \bar{x}^i)$ with an invariant optimal harvesting policy: harvest x^i and sow exactly the same area to keep an invariant configuration. It is also clear that we must have $\bar{x}^i = 0$ since otherwise a policy that harvests a little more at time $t = 0$ and x^i in all other periods would provide a greater benefit contradicting optimality. Since the area constraint imposes $\sum_{i\in I} n_i x^i = S$, we are left with only $k-1$ degrees of freedom. For the rest of this paper, we denote $\sigma_i = b^{n_i}/(1-b^{n_i})$ and without loss of generality, we assume that the species are ordered in such a way that $\sigma_1 U_1'(0) \geq \sigma_2 U_2'(0) \geq \cdots \geq \sigma_k U_k'(0)$.

In the sequel, we denote $\mathbb{X}^* \in \Delta^0$ the state of the previous form with $\bar{x}^i = 0$ and $x^i = x^{*i}$, where x^* is the unique optimal solution of the strictly convex program

$$(S) \quad \begin{cases} \max \ \sum_{i\in I} n_i \sigma_i U_i(x^i), \\ \text{s.t.} \ \ x^i \geq 0 \text{ and } \sum_{i\in I} n_i x^i = S. \end{cases}$$

Denote $I^* = \{i \in I: x^{*i} > 0\}$ the species that are present in $\mathbb{X}^*$ and let r be the Lagrange multiplier of the area constraint $\sum_{i\in I} n_i x^i = S$, so that the optimal solution is characterized by $\sigma_i U_i'(x^{*i}) = r$ for $i \in I^*$ and $\sigma_j U_j'(0) \leq r$ for $j \notin I^*$. The ordering of the species and the strict concavity of U_i then imply that whenever $x^{*i} > 0$ we must also have $x^{*j} > 0$ for all $j < i$, so that $I^* = \{1, \ldots, i^*\}$ for some index i^*. This leads to a constructive method for solving (S) in which the values $x^1, x^2, \ldots$ are increased sequentially:

- increase x^1 as much as possible until $\sigma_1 U_1'(x^1)$ decreases to the value $\sigma_2 U_2'(0)$.
- continue increasing x^1 and x^2 simultaneously preserving the equality $\sigma_1 U_1'(x^1) = \sigma_2 U_2'(x^2)$ until this common value decreases to the level $\sigma_3 U_3'(0)$.
- continue increasing x^1, x^2, x^3 keeping the equality $\sigma_1 U_1'(x^1) = \sigma_2 U_2'(x^2) = \sigma_3 U_3'(x^3)$ until this common value hits the level $\sigma_4 U_4'(0)$.
- continue this procedure with $x^4, x^5, \ldots, x^{i^*}$ stopping as soon as $\sum_{i=1}^{i^*} n_i x^i = S$.

The point found by this procedure satisfies the optimality conditions for (S), and it is therefore the unique optimal solution. More interestingly, we have

Theorem 2.1. *The state $\mathbb{X}^*$ defined above is the unique sustainable state.*

Proof. Let us first show that the stationary trajectory

$$x_t^i = u_t^i = x^{*i} \text{ for all } t \qquad \text{and} \qquad \bar{\mathbf{x}}^i = 0 \tag{4}$$

is optimal for $P(\mathbb{X}^*)$. To this end, it suffices to consider the Lagrangian

$$\begin{aligned} L = &\sum_{i\in I}\left\{\sum_{t=0}^{\infty} b^t U_i(u_t^i) + \sum_{t=0}^{\infty}\mu_t^i u_t^i + \sum_{t=1}^{\infty}\bar{\lambda}_t^i \bar{x}_t^i + \sum_{t=n_i}^{\infty}\lambda_t^i x_t^i\right\} \\ &+ \sum_{i\in I}\sum_{t=0}^{\infty}\alpha_t^i(\bar{x}_t^i + x_t^i - u_t^i - \bar{x}_{t+1}^i) + \sum_{t=0}^{\infty}\left[\theta_t \sum_{i\in I}(u_t^i - x_{t+n_i}^i)\right] \end{aligned} \tag{5}$$

together with the following set of l^1-multipliers

$$\begin{cases} \mu_t^i = 0, \\ \theta_t = b^t r, \\ \alpha_t^i = b^t[r + U_i'(x^{*i})], \\ \lambda_t^i = b^t\left[\dfrac{r}{\sigma_i} - U_i'(x^{*i})\right], \\ \bar{\lambda}_t^i = \alpha_t^i(1-b)/b, \end{cases} \tag{6}$$

where r is as before the Lagrange multiplier corresponding to the area constraint in (S). Observe that L is concave with respect to the primal variables $(\mathbf{x}^i, \bar{\mathbf{x}}^i, \mathbf{u}^i)$. To verify the stationarity of L, we must show that $\nabla L = 0$ at the point given by (4) with multipliers given by (6). Let us show that the partial derivatives[1] $L_{u_t^i} = L_{x_t^i} = L_{\bar{x}_t^i} = 0$ for all $i \in I$ and $t \in \mathbb{N}$. Namely, a straightforward computation yields

$$L_{u_t^i} = b^t U_i'(u_t^i) + \mu_t^i - \alpha_t^i + \theta_t = b^t U_i'(x^{*i}) + 0 - b^t[r + U_i'(x^{*i})] + b^t r = 0,$$

$$L_{x_t^i} = \lambda_t^i + \alpha_t^i - \theta_{t-n_i} = b^t\left[\frac{r}{\sigma_i} - U_i'(x^{*i})\right] + b^t[r + U_i'(x^{*i})] - b^{t-n_i} r = b^t r\left[\frac{1}{\sigma_i} + 1 - \frac{1}{b^{n_i}}\right] = 0,$$

$$L_{\bar{x}_t^i} = \bar{\lambda}_t^i + \alpha_t^i - \alpha_{t-1}^i = \frac{\alpha_t^i}{b} - \alpha_t^i + \alpha_t^i - \alpha_{t-1}^i = 0.$$

Moreover, since we can see directly that the complementary slackness is satisfied and that θ, α^i, $\bar{\lambda}^i$, λ^i are all in l_+^1, we conclude that the proposed trajectory is a stationary point of the Lagrangian, hence a solution to $P(\mathbb{X}^*)$. This proves that $\mathbb{X}^*$ is sustainable.

To prove the converse, let $\mathbb{X}$ be a sustainable state with $X^i = (x^i, \ldots, x^i, 0)$. We claim that when $x^i > 0$, then $\sigma_i U_i'(x^i) \geq \sigma_j U_j'(x^j)$ for all $j \in I$. Indeed, let us perturb the optimal harvesting policy as follows: at time $t = 0$, we sow $x^i - \epsilon$ and $x^j + \epsilon$ instead of x^i and x^j, while in all subsequent periods, we harvest all mature trees and sow the harvested areas with the same species they had. The benefit V_ϵ derived from this perturbed policy must be less than the value V obtained with the optimal one, which gives

$$V_\epsilon - V = \frac{b^{n_i}}{1-b^{n_i}}[U_i(x^i - \epsilon) - U_i(x^i)] + \frac{b^{n_j}}{1-b^{n_j}}[U_j(x^j + \epsilon) - U_j(x^j)] \leq 0.$$

Dividing by ϵ and letting $\epsilon \to 0$, we deduce $\sigma_i U_i'(x^i) \geq \sigma_j U_j'(x^j)$ as claimed. Setting $I^* = \{i\colon x^i > 0\}$, it follows that $\sigma_i U_i'(x^i)$ is constant for $i \in I^*$ and larger than the value of this expression for $i \notin I^*$. This implies that the vector $(x^i)_{i\in I}$ is an optimal solution for (S), so that $x^i = x^{*i}$ completing the proof. ☐

Remark 2.1. As mentioned in Remark 1.1, our model restricts the harvests to trees beyond some exogenously fixed maturity ages n_i. A possible choice is to take n_i equal to the Faustmann [2] rotation age of the corresponding species (which depends on the discount factor b). In such a case, it can be shown that the optimally invariant state for the Mitra and Wan model [8], which allows harvesting at all ages, coincides with the sustainable state of our model. For a proof of this, we refer to the recent thesis by Piazza [11].

[1] All derivatives of L are evaluated at the point defined by (4) and (6). For the sake of simplicity, we do not write the evaluation point every time.

2.2. GPCs. For a one-species forest, it was proved in Rapaport et al. [12] that after a finite time, every optimal trajectory becomes *greedy* in the sense that all mature trees are harvested, and then the evolution becomes periodic. For multiple-species, this notion of greedy trajectory is incomplete because it does not specify a re-sowing policy. A typical example of a *greedy* evolution for two species with $n_1 = 3$ and $n_2 = 2$ would be

$$\begin{pmatrix} \bar{x}_0^1 \\ x_0^1 \\ x_1^1 \\ x_2^1 \end{pmatrix} \begin{pmatrix} \bar{x}_0^2 \\ x_0^2 \\ x_1^2 \end{pmatrix} \to \begin{pmatrix} 0 \\ x_1^1 \\ x_2^1 \\ s \end{pmatrix} \begin{pmatrix} 0 \\ x_1^2 \\ t \end{pmatrix},$$

with $s + t = \bar{x}_0^1 + x_0^1 + \bar{x}_0^2 + x_0^2$. We will refer to a *greedy periodic strategy* when the new seedlings preserve the area just harvested for each species, i.e., $s = \bar{x}_0^1 + x_0^1$ and $t = \bar{x}_0^2 + x_0^2$.

Definition 2.2. A feasible trajectory $(\mathbf{x}^i, \bar{\mathbf{x}}^i, \mathbf{u}^i)_{i\in I}$ is called *greedy* if $\bar{x}_t^i = 0$ for all $i \in I$ and $t \in \mathbb{N}$. It will be called a GPC if in addition $x_{t+n_i}^i = u_t^i$ for all $i \in I$ and $t \in \mathbb{N}$. We denote Δ^g the set of initial states $\mathbb{X}_0 \in \Delta^0$ for which there exists a greedy optimal trajectory, and Δ^p those having an optimal trajectory that is a GPC.

As previously noted, the discounting of future benefits favors the earlier harvests and suggests the advantage of greedy strategies. This is balanced, however, by the concavity of the utilities, so that a transient period may exist in which some trees are kept beyond maturity to transfer area between different age classes and species. In §3, we will describe two situations in which these area transfers occur only during an initial phase after which every optimal trajectory becomes greedy. On the other hand, it is worth mentioning that since the optimal harvests are uniquely determined, the area balance implies that an optimal trajectory issued from an initial state $\mathbb{X}_0 \in \Delta^g$ must be unique; namely, a greedy one, and therefore Δ^g is forward invariant through optimal policies: $\mathbb{X}_t \in \Delta^g \Rightarrow \mathbb{X}_{t+1} \in \Delta^g$. In the case of a GPC, the dynamics of all the species are uncoupled producing cyclic trajectories of period n_i for species i, so that the sequence of states $(\mathbb{X}_t)_{t\in\mathbb{N}}$ is cyclic with period equal to the least common multiple N of the maturity ages $n_1, \ldots, n_k$.

Clearly, the sustainable state $\mathbb{X}^*$ yields an optimal GPC, so that $\mathbb{X}^* \in \Delta^p$. In §2.3, we will show that under appropriate conditions, this is, in fact, the only point in Δ^p. Let us first characterize this set.

Theorem 2.2. *Let $\mathbb{X}_0 \in \Delta^0$ and consider the periodic sequences $(x_t^i)_{t\in\mathbb{N}}$ built from $\mathbb{X}_0$ with $x_{t+n_i}^i = x_t^i$. Then $\mathbb{X}_0 \in \Delta^p$ iff for all $i, j \in I$, and $t \in \mathbb{N}$, we have*[2]

$$\begin{aligned} &\text{(a)}\ \ U_i'(x_t^i) \ge b\, U_i'(x_{t+1}^i), \\ &\text{(b)}\ \ x_t^i > 0 \ \Rightarrow\ \sigma_i U_i'(x_t^i) \ge b^{n_j}[\sigma_i U_i'(x_{t+n_j}^i) + U_j'(x_t^j)]. \end{aligned} \tag{7}$$

In the proof, we will exploit the following useful consequence of the latter condition.

Lemma 2.1. *Consider the periodic sequences $(x_t^i)_{t\in\mathbb{N}}$ with $x_{t+n_i}^i = x_t^i$. If condition (7b) holds, then*

$$x_t^i > 0 \ \Rightarrow\ \sigma_i U_i'(x_t^i) \ge \sigma_j U_j'(x_t^j). \tag{8}$$

Proof. We note that if the conclusion in (7b) holds at time t, it does so at $t + n_j$ even if $x_{t+n_j}^i = 0$. Indeed, suppose by contradiction that the conclusion in (7b) holds at time t; that is,

$$\sigma_i U_i'(x_t^i) \ge b^{n_j}[\sigma_i U_i'(x_{t+n_j}^i) + U_j'(x_t^j)],$$

but not at time $t + n_j$ (in which case we must have $x_{t+n_j}^i = 0$),

$$b^{n_j}[\sigma_i U_i'(x_{t+2n_j}^i) + U_j'(x_t^j)] > \sigma_i U_i'(x_{t+n_j}^i)$$

(here we use $x_{t+n_j}^j = x_t^j$). Adding these two inequalities yields

$$\sigma_i[U_i'(x_t^i) + b^{n_j} U_i'(x_{t+2n_j}^i)] > \sigma_i(1 + b^{n_j}) U_i'(0),$$

contradicting the fact that $U_i'(\cdot)$ is decreasing.

[2] Notice that it suffices to check condition (a) for $t = 0, 1, \ldots, n_i - 1$ while for condition (b), it suffices to check for $t = 0, 1, \ldots, n_{ij} - 1$, where n_{ij} is the least common multiple of the maturity ages n_i and n_j.

Now, if $x_t^i > 0$, then it follows inductively that (7b) holds for all integers of the form $t + ln_j$ with $l \in \mathbb{N}$. Writing down these inequalities for $l = 0, 1, \ldots, n_i - 1$, we get

$$\begin{aligned} \sigma_i U_i'(x_t^i) &\geq b^{n_j}[\sigma_i U_i'(x_{t+n_j}^i) + U_j'(x_t^j)] \\ b^{n_j}\sigma_i U_i'(x_{t+n_j}^i) &\geq b^{2n_j}[\sigma_i U_i'(x_{t+2n_j}^i) + U_j'(x_{t+n_j}^j)] = b^{2n_j}[\sigma_i U_i'(x_{t+2n_j}^i) + U_j'(x_t^j)] \\ &\vdots \\ b^{(n_i-1)n_j}\sigma_i U_i'(x_{t+(n_i-1)n_j}^i) &\geq b^{n_i n_j}[\sigma_i U_i'(x_{t+n_i n_j}^i) + U_j'(x_{t+(n_i-1)n_j}^j)] = b^{n_i n_j}[\sigma_i U_i'(x_{t+n_i n_j}^i) + U_j'(x_t^j)], \end{aligned}$$

which added together yield

$$\sigma_i U_i'(x_t^i) \geq b^{n_i n_j}\sigma_i U_i'(x_t^i) + (b^{n_j} + b^{2n_j} + \cdots + b^{n_i n_j})U_j'(x_t^j).$$

This inequality may be equivalently rewritten as

$$(1 - b^{n_i n_j})\sigma_i U_i'(x_t^i) \geq \frac{1 - b^{n_i n_j}}{1 - b^{n_j}} b^{n_j} U_j'(x_t^j),$$

which readily gives $\sigma_i U_i'(x_t^i) \geq \sigma_j U_j'(x_t^j)$. □

Proof of Theorem 2.2 To prove the necessity of condition (7), take $\mathbb{X}_0 \in \Delta^p$ such that the periodic sequences $(x_t^i)_{t\in\mathbb{N}}$ are optimal for $P(\mathbb{X}_0)$.

If $x_t^i = 0$, property (7a) is obvious since U_i' is decreasing. Suppose next that $x_t^i > 0$ and perturb the GPC by harvesting $x_t^i - \epsilon$ at stage t keeping ϵ for the next period as overmature trees, and then harvest $x_{t+1}^i + \epsilon$ at stage $t+1$ after which we continue with a GPC. This modification must give a smaller benefit than the optimal GPC, from which we get

$$b^t \frac{1}{1 - b^{n_i}}[U_i(x_t^i - \epsilon) - U_i(x_t^i)] + b^{t+1}\frac{1}{1 - b^{n_i}}[U_i(x_{t+1}^i + \epsilon) - U_i(x_{t+1})] \leq 0,$$

which divided by ϵ and letting it to 0 yields (7a).

A similar argument proves (7b). The idea is now as follows: at stage t, we sow $x_t^i - \epsilon$ instead of x_t^i on species i and transfer this ϵ to species j. Then, at stage $t + n_j$, we harvest this ϵ from species j and return it to species i, modifying the sowing at that stage. In all the other stages, we use a greedy periodic strategy. Noting that $x_{t+n_j}^j = x_t^j$, the difference of benefit is given by

$$b^t \frac{b^{n_i}}{1 - b^{n_i}}[U_i(x_t^i - \epsilon) - U_i(x_t^i)] + b^{t+n_j}\frac{b^{n_i}}{1 - b^{n_i}}[U_i(x_{t+n_j}^i + \epsilon) - U_i(x_{t+n_j}^i)] + b^{t+n_j}[U_j(x_t^j + \epsilon) - U_j(x_t^j)] \leq 0,$$

and the conclusion follows as before dividing by ϵ and letting it to 0.

Let us establish next the sufficiency. Take $\mathbb{X}_0$ satisfying (7) and consider the corresponding GPC. To prove its optimality, it suffices to check that it is a stationary point for the Lagrangian (5) with the following multipliers:

$$\begin{cases} \mu_t^i = 0, \\ \theta_t = b^t \sigma_1 U_1'(x_t^1), \\ \lambda_t^i = b^t \left\{ \sigma_1 \left[\dfrac{1}{b^{n_i}} U_1'(x_{t-n_i}^1) - U_1'(x_t^1) \right] - U_i'(x_t^i) \right\}, \\ \alpha_t^i = \theta_t + b^t U_i'(x_t^i), \\ \bar{\lambda}_t^i = \alpha_{t-1}^i - \alpha_t^i. \end{cases}$$

All these multipliers are of the form b^t multiplied by some bounded sequence so they belong to l^1, while their nonnegativity is evident except for λ_t^i and $\bar{\lambda}_t^i$. The inequality $\bar{\lambda}_t^i \geq 0$ follows directly from (7a). For $\lambda_t^i \geq 0$, we observe that this is ensured by condition (7b) whenever $x_{t-n_i}^1 > 0$, while when $x_{t-n_i}^1 = 0$ using the monotonicity of U_i' and the fact that $\sigma_1 U_1'(0) \geq \sigma_i U_i'(0)$, we get

$$\sigma_1 \left[\frac{1}{b^{n_i}} U_1'(0) - U_1'(x_t^1) \right] - U_i'(x_t^i) \geq \sigma_1 \left[\frac{1}{b^{n_i}} - 1 \right] U_1'(0) - U_i'(x_t^i) \geq U_i'(0) - U_i'(x_t^i) \geq 0,$$

which shows that all the multipliers belong to l_+^1.

Verification of stationarity is analogous to that of Theorem 2.1 and only the verification of the complementary slackness for the constraint $x_t^i \geq 0$ is not obvious. Let us show that $x_t^i > 0 \Rightarrow \lambda_t^i = 0$. Indeed, when $x_t^i > 0$, we know from Lemma 2.1 that $\sigma_i U_i'(x_t^i) \geq \sigma_1 U_1'(x_t^1)$. Now, if $x_t^1 > 0$ the same argument yields $\sigma_1 U_1'(x_t^1) \geq \sigma_i U_i'(x_t^i)$, while when $x_t^1 = 0$, we have $\sigma_1 U_1'(x_t^1) = \sigma_1 U_1'(0) \geq \sigma_i U_i'(0) \geq \sigma_i U_i'(x_t^i)$. Hence, in all cases, we have $\sigma_i U_i'(x_t^i) = \sigma_1 U_1'(x_t^1)$. A similar reasoning yields $\sigma_i U_i'(x_t^i) = \sigma_1 U_1'(x_{t-n_i}^1)$, which readily implies $\lambda_t^i = 0$. This completes the proof of optimality of the GPC for $P(\mathbb{X}_0)$. □

Theorem 2.2 may also be used to characterize the optimal GPCs for a one-species forest. Indeed, introducing an auxiliary annual second species with benefit $U_2 \equiv 0$ (note that in the previous proof, we did not use strict monotonicity nor strict concavity of the U_i's), we get

Corollary 2.1. *For a one-species forest, the initial state $X^0 = (x_{n-1}, \ldots, x_0, 0)$ produces an optimal GPC if and only if $U'(x_t) \geq bU'(x_{t+1})$ for all $t = 0, \ldots, n-1$.*

2.3. Relation between the GPCs and the sustainable state. We already noted that $\mathbb{X}^* \in \Delta^p$. We will prove that $\mathbb{X}^*$ is, in fact, the only element in Δ^p whenever the maturity ages of the species that are present in $\mathbb{X}^*$, are relatively primes. Let us define the *support* of a given state $\mathbb{X}_0 \in \Delta^p$ as the set of indices $I(\mathbb{X}_0) = \{i \in I: X^i \neq 0\}$.

Lemma 2.2. *If $\mathbb{X}_0 \in \Delta^p$, then $I(\mathbb{X}_0) = \{1, \ldots, i_0\}$ for some $i_0 \geq i^*$.*

Proof. We observe that if $X^i \equiv 0$, then $X^j \equiv 0$ for all $j > i$, which implies that $I(\mathbb{X}_0) = \{1, \ldots, i_0\}$. This first statement is a direct consequence of Lemma 2.1, the strict concavity of the functions U_i, and the order in which species are numbered so that $\sigma_i U_i'(0)$ decreases with i. Now, to prove that $i_0 \geq i^*$, we proceed by contradiction assuming that $i_0 < i^*$. The area balance

$$\sum_{i \in I_0} \sum_{t=0}^{n_i - 1} x_t^i = S = \sum_{i \in I^*} \sum_{t=0}^{n_i - 1} x^{i*}$$

implies that there exist $i \leq i_0$ and $t \in \mathbb{N}$ such that $x_t^i > x^{i*}$, and thus

$$\sigma_i U_i'(x_t^i) < \sigma_i U_i'(x^{i*}) = \sigma_j U_j'(x^{j*}) \quad \forall\, j \in I^*.$$

In particular, taking $j \in I^* \backslash I_0$ and using Lemma 2.1, we reach the contradiction

$$\sigma_i U_i'(x_t^i) \geq \sigma_j U_j'(0) > \sigma_j U_j'(x^{*j}) > \sigma_i U_i'(x_t^i). \quad \square$$

For the sequel, we denote $\gcd(n_1, \ldots, n_i)$ the greatest common divisor of the integers $n_1, \ldots, n_i$, and we recall the following property related to the Chinese Remainder Theorem.

Lemma 2.3. *If $v \equiv u \bmod \gcd(n, m)$, then there exists $t \in \mathbb{N}$ such that $t \equiv u \bmod n$ and $t \equiv v \bmod m$.*

Theorem 2.3. *If $\gcd(n_1, \ldots, n_{i*}) = 1$, then $\Delta^p = \{\mathbb{X}^*\}$. Otherwise, $\gcd(n_1, \ldots, n_{i_0}) > 1$ for all $\mathbb{X}_0 \in \Delta^p$.*

Proof. Using Lemma 2.2, it can be easily checked that the result reduces to showing that any point $\mathbb{X}_0 \in \Delta^p$ with $\gcd(n_1, \ldots, n_{i_0}) = 1$ satisfies $\mathbb{X}_0 = \mathbb{X}^*$. To prove the latter, let us fix u with $x_u^{i_0} > 0$.

For each $v \equiv u \bmod \gcd(n_{i_0}, n_{i_0 - 1})$, we may take t as given by Lemma 2.3 and, since $\mathbb{X}_0 \in \Delta^p$, the periodicity of the optimal trajectories implies $x_t^{i_0} = x_u^{i_0}$ and $x_t^{i_0 - 1} = x_v^{i_0 - 1}$. Then, invoking Theorem 2.2 and Lemma 2.1, we may use (8) to get

$$\sigma_{i_0} U_{i_0}'(x_u^{i_0}) \geq \sigma_{i_0 - 1} U_{i_0 - 1}'(x_v^{i_0 - 1}).$$

This implies, in turn, that $x_v^{i_0 - 1} > 0$ since otherwise we reach the contradiction

$$\sigma_{i_0} U_{i_0}'(x_u^{i_0}) \geq \sigma_{i_0 - 1} U_{i_0 - 1}'(0) \geq \sigma_{i_0} U_{i_0}'(0) > \sigma_{i_0} U_{i_0}'(x_u^{i_0}),$$

and therefore using (8) once again, we deduce

$$x_v^{i_0 - 1} > 0 \quad \text{and} \quad \sigma_{i_0} U_{i_0}'(x_u^{i_0}) = \sigma_{i_0 - 1} U_{i_0 - 1}'(x_v^{i_0 - 1}). \tag{9}$$

Similarly, if $w \equiv u \bmod \gcd(n_{i_0}, n_{i_0 - 1}, n_{i_0 - 2})$, we may use Lemma 2.3 to find $v \in \mathbb{N}$ such that $v \equiv u \bmod \gcd(n_{i_0}, n_{i_0 - 1})$ and $v \equiv w \bmod n_{i_0 - 2}$. From the previous analysis, the former congruence implies (9) while, thanks to the periodicity of the sequence $(x_t^{i_0 - 2})$, the second one implies $x_v^{i_0 - 2} = x_w^{i_0 - 2}$, and therefore using (8) again, we get

$$\sigma_{i_0 - 1} U_{i_0 - 1}'(x_v^{i_0 - 1}) \geq \sigma_{i_0 - 2} U_{i_0 - 2}'(x_w^{i_0 - 2}).$$

As before, we cannot have $x_w^{i_0-2}=0$ and we deduce

$$x_w^{i_0-2}>0 \qquad \text{and} \qquad \sigma_{i_0}U'_{i_0}(x_u^{i_0})=\sigma_{i_0-2}U'_{i_0-2}(x_w^{i_0-2}).$$

Proceeding inductively, we conclude that for all $t\equiv u \bmod \gcd(n_{i_0},\ldots,n_1)$, we have

$$x_t^1>0 \qquad \text{and} \qquad \sigma_{i_0}U'_{i_0}(x_u^{i_0})=\sigma_1 U'_1(x_t^1),$$

and since $\gcd(n_1,\ldots,n_{i_0})=1$, it follows that $x_t^1\equiv x^1$ is constant for all $t\in\mathbb{N}$. To establish the theorem, it remains to prove that

$$\sigma_i U'_i(x_t^i)=\sigma_1 U'_1(x^1) \quad \forall\, i\in I(\mathbb{X}_0),\quad t\in\mathbb{N} \tag{10}$$

since this clearly forces $\mathbb{X}_0$ to be the sustainable state $\mathbb{X}^*$. We already know from Lemma 2.1 that (10) holds when $x_t^i>0$. We finish the proof by noting that for each $i\in I(\mathbb{X}_0)$, there exists at least one $t\in\mathbb{N}$ with $x_t^i>0$, and then there cannot be an element $x_u^i=0$, since this would yield the contradiction

$$\sigma_1 U'_1(x^1)\geq \sigma_i U'_i(0)>\sigma_i U'_i(x_t^i)=\sigma_1 U'_1(x^1). \quad \square$$

3. Convergence of optimal trajectories.

We are now ready to discuss the long-term behavior of the optimal harvesting policies. The previous sections described some special states from which the optimal trajectory is either invariant or periodic. We claim that such behavior is typical in the sense that an optimally managed forest converges either to the sustainable state or to an optimal GPC. To prove this *global attractor property*, we first establish conditions under which all optimal trajectories become greedy after a finite time, and then we rely on a suitable Lyapunov function to analyze their asymptotic behavior. Finally, we show that the result holds not only for greedy trajectories but for every optimal harvesting policy.

3.1. Optimal trajectories become greedy.

We start by giving conditions under which every optimal strategy becomes greedy after finitely many steps. Throughout we let $\mathbf{x}^i$, $\bar{\mathbf{x}}^i$, $\mathbf{u}^i$ be an optimal trajectory for $P(\mathbb{X}_0)$, while $X_t^i=(x_{t+n_i-1}^i,\ldots,x_t^i,\bar{x}_t^i)$ stands for the corresponding state of the i-th species in the forest at time t and $\mathbb{X}_t=(X_t^1,\ldots,X_t^k)$. Our first result concerns the case of an annual alternative use of the land such as agriculture or rent.

Proposition 3.1. *If there is an annual species with maturity age $n_d=1$, then $\mathbb{X}_t\in\Delta^g$ for all $t\geq 2\bar{n}-1$, where $\bar{n}=\max_{i\in I} n_i$.*

Proof. Since $n_d=1$, there is no gain in postponing the harvest of mature areas for that species: harvesting all of $\bar{x}_t^d$ and resowing it immediately with the same species provides a feasible trajectory that yields a strictly larger benefit, which contradicts optimality. Hence $\bar{x}_t^d=0$ for $t\geq 1$.

For the species $i\neq d$, the argument is a bit more elaborated.

First, note that in each interval of length n_i such as $p+1,\ldots,p+n_i$ there is at least one $\bar{x}_t^i=0$. Indeed, if this was not the case, then at time p, we could harvest a small additional area $\epsilon>0$ and resow it immediately as species i, modifying the trajectory as $\bar{x}_t^i-\epsilon$ for $t=p+1,\ldots,p+n_i$ and $x_{p+n_i}^i+\epsilon$, after which we rejoin the original optimal strategy. This modified trajectory would increase the benefit by an amount $b^p[U_i(u_p^i+\epsilon)-U_i(u_p^i)]>0$, contradicting optimality.

Next, observe that for $t\geq n_i$, we have $\bar{x}_t^i=0\Rightarrow \bar{x}_{t+1}^i=0$. To see this, we proceed again by contradiction: if $\bar{x}_t^i=0<\bar{x}_{t+1}^i$, then $u_t^i<\bar{x}_t^i+x_t^i=x_t^i$, which means that at stage t, we do not harvest all the available trees of species i. If we backtrack to stage $t-n_i$ when x_t^i was sown, we could have taken out a small area $\epsilon>0$ and sow it as species d making an additional benefit in the next period of $b^{t-n_i+1}[U_d(u_{t-n_i+1}^d+\epsilon)-U_d(u_{t-n_i+1}^d)]>0$, after which this ϵ is returned to species i, so that at stage $t+1$ the trees reaching maturity $x_{t+1}^i+\epsilon$ compensate the loss of overmature trees $\bar{x}_{t+1}^i-\epsilon$. This trajectory allows to harvest the same areas as in the original strategy, except at stage $t-n_i+1$, where we make an extra benefit contradicting optimality.

Combining the previous properties, we may conclude: in the interval $n_i,\ldots,2n_i-1$, there exists at least one t such that $\bar{x}_t^i=0$, condition that must hold thereafter. $\square$

Remark 3.1. The previous result is sharp since there are examples with $\bar{x}_{2n_i-2}^i>0$ such as a two-species forest initially composed only of young trees: $X_0^1=(S,0,\ldots,0)$ and $X_0^2=(0,0)$. We must wait n_1 periods to have available trees to harvest, and then the optimal strategy may consist in harvesting and sowing only a fraction of them, keeping the rest as overmature trees to reshape the forest with a more balanced age-class distribution. This process may take up to n_1-1 additional periods to reach the regime $\bar{x}_t^1\equiv 0$.

We could not prove, in general, that the optimal trajectories become greedy when $n_i > 1$ for all i. However, we can give the following sufficient condition, which holds, in particular, when the benefit functions U_i are "almost" linear or when either b or S are small.

PROPOSITION 3.2. *If $U_i'(S) \geq b\, U_i'(0)$ for all i, then $\mathbb{X}_t \in \Delta^g$ for all $t \geq 1$.*

PROOF. We must show that $\bar{x}_t^i = 0$ for all $i \in I$ and $t \geq 1$. Given that all the cases are symmetric, we just prove it for $i = 1$. The argument in the previous proof shows that in any interval of length n_1, there is at least one $\bar{x}_t^1 = 0$, so we may find t as large as we want with $\bar{x}_t^1 = 0$. The conclusion will follow by backward induction if we show that $\bar{x}_{t+1}^1 = 0 \Rightarrow \bar{x}_t^1 = 0$ for $t \geq 1$.

To prove the latter, we proceed by contradiction: suppose $\bar{x}_t^1 > \bar{x}_{t+1}^1 = 0$, so that $u_t^1 = \bar{x}_t^1 + x_t^1 > 0$ and then there is at least one $i \in I$ such that $x_{t+n_i}^i > 0$. We take $\epsilon = \min\{\bar{x}_t^1, x_{t+n_i}^i\} > 0$ and consider the following perturbed trajectory: at time $t-1$, we harvest $u_{t-1}^1 + \epsilon$ instead of u_{t-1}^1 of species 1 (we can do it because $\bar{x}_t^1 > \epsilon$) and we transfer this ϵ to species i sowing $x_{t+n_i-1}^i + \epsilon$ (harvesting of species i is not altered, hence at time $t+n_i$, we have $\bar{x}_{t+n_i}^i + \epsilon$); at time t, we harvest ϵ less of species 1, $u_t^1 - \epsilon$, and we plant ϵ less of species i, $x_{t+n_i}^i - \epsilon$. All other variables remain untouched. Observe that at times $t+n_i-1$ and $t+n_i$, Equation (1) holds without changing the harvesting: $\bar{x}_{t+n_i}^i + \epsilon = \bar{x}_{t+n_i-1}^i + x_{t+n_i-1}^i + \epsilon - u_{t+n_i-1}^i$ and $\bar{x}_{t+n_i+1}^i = \bar{x}_{t+n_i}^i + \epsilon + x_{t+n_i}^i - \epsilon - u_{t+n_i}^i$, implying that the change in sowing does not affect the benefit. The new trajectory is also feasible and strict concavity of U_1 implies that the benefit difference is

$$\begin{aligned} V_\epsilon - V &= b^{t-1}[U_1(u_{t-1}^1 + \epsilon) + b\, U_1(u_t^1 - \epsilon) - U_1(u_{t-1}^1) - b\, U_1(u_t^1)] \\ &> b^{t-1}\epsilon[U_1'(u_{t-1}^1 + \epsilon) - b\, U_1'(u_t^1 - \epsilon)], \end{aligned}$$

so that $V_\epsilon - V > b^{t-1}\epsilon[U_1'(S) - b\, U_1'(0)] \geq 0$, contradicting optimality. □

3.2. Lyapunov function. In §3.1, we described conditions under which an optimally managed forest attains the set Δ^g in finite time. Since this set is forward invariant under an optimal strategy, the trajectory remains in this set thereafter: $\mathbb{X}_t \in \Delta^g \Rightarrow \mathbb{X}_{t+1} \in \Delta^g$. This does not prove, however, that an optimal policy will necessarily lead to a GPC nor to the sustainable state, because we have not yet checked that the sown areas coincide with the harvests. To address this issue, we will introduce a Lyapunov function Φ: $\Delta^0 \to \mathbb{R}$ with the property that any optimal sequence $\mathbb{X}_t$ becomes an "N-step" uphill strategy as soon as it enters the set Δ^g; namely, we have the implication $\mathbb{X}_t \in \Delta^g \Rightarrow \Phi(\mathbb{X}_{t+N}) \geq \Phi(\mathbb{X}_t)$ for a specific $N \in \mathbb{N}$. This property will be used later on to establish that every optimal trajectory converges asymptotically toward the set Δ^p. Strictly speaking, Φ is not a Lyapunov function since it does not increase at every stage. We may recover a standard Lyapunov function by considering the sum or the maximum over N consecutive periods, however, this would make the arguments unnecessarily obscure.

Let us consider the map

$$\Phi(\mathbb{X}_0) = G(\mathbb{X}_0) - \sum_{i\in I}\sum_{t=0}^{n_i-2} \frac{b^t - b^{n_i-1}}{1-b^{n_i}} U_i(x_t^i), \tag{11}$$

where $G(\mathbb{X}_0)$ is the optimal benefit obtained from state $\mathbb{X}_0$ by using a greedy policy; that is,

$$\begin{aligned} G(\mathbb{X}_0) = \max\ & \sum_{i\in I}\sum_{t=0}^{\infty} b^t U_i(x_t^i), \\ \text{s.t.}\ & \mathbf{x}^i \in l_+^\infty, \\ & \sum_{i\in I} x_{t+n_i}^i = \sum_{i\in I} x_t^i. \end{aligned}$$

Naturally, when restricted to states in Δ^g, the latter coincides with the optimal value of $P(\mathbb{X}_0)$. The clue for the subsequent asymptotic analysis is the following property.

THEOREM 3.1. *Let N be the least common multiple of $n_1, \ldots, n_k$. Then, for all $\mathbb{X}_0 \in \Delta^g$, we have*

$$\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0)$$

with strict inequality unless $\mathbb{X}_0 \in \Delta^p$.

Proof. To simplify the notation set $G_t = G(\mathbb{X}_t)$ and denote

$$P_t = \sum_{i\in I} \frac{1}{1-b^{n_i}} \sum_{j=t}^{t+n_i-1} b^{j-t} U_i(x_j^i),$$

the benefit of a GPC started from state $\mathbb{X}_t$. Since G_t is the optimal greedy benefit, we have $G_t \geq P_t$, which can be written as $(1-b^N)G_t \geq (1-b^N)P_t$ and then

$$G_t \geq b^N G_t + \sum_{i\in I} \frac{1-b^N}{1-b^{n_i}} \sum_{j=t}^{t+n_i-1} b^{j-t} U_i(x_j^i).$$

Now, Bellman's principle of dynamic programming gives

$$G_0 = \sum_{i\in I}\sum_{j=0}^{t-1} b^j U_i(x_j^i) + b^t G_t,$$

$$G_t = \sum_{i\in I}\sum_{j=t}^{N-1} b^{j-t} U_i(x_j^i) + b^{N-t} G_N,$$

which when plugged into the previous inequality yields

$$b^{N-t}G_N + \sum_{i\in I}\sum_{j=t}^{N-1} b^{j-t}U_i(x_j^i) \geq b^{N-t}G_0 - \sum_{i\in I}\sum_{j=0}^{t-1} b^{N-t+j}U_i(x_j^i) + \sum_{i\in I}\frac{1-b^N}{1-b^{n_i}}\sum_{j=t}^{t+n_i-1} b^{j-t}U_i(x_j^i).$$

Adding up these equations for $t = 0, 1, \ldots, N-1$, we get

$$\frac{b(1-b^N)}{1-b}G_N + \sum_{t=0}^{N-1}\sum_{i\in I}\sum_{j=t}^{N-1} b^{j-t}U_i(x_j^i) \;\geq\; \frac{b(1-b^N)}{1-b}G_0 - \sum_{t=0}^{N-1}\sum_{i\in I}\sum_{j=0}^{t-1} b^{N-t+j}U_i(x_j^i) + \sum_{t=0}^{N-1}\sum_{i\in I}\frac{1-b^N}{1-b^{n_i}}\sum_{j=t}^{t+n_i-1} b^{j-t}U_i(x_j^i), \tag{12}$$

and exchanging the order in all these multiples summations, we deduce

$$\begin{aligned}
&\frac{b(1-b^N)}{1-b}G_N + \sum_{j=0}^{N-1}\frac{1-b^{j+1}}{1-b}\sum_{i\in I}U_i(x_j^i)\\
&\geq \frac{b(1-b^N)}{1-b}G_0 - \sum_{j=0}^{N-1}\frac{b^{j+1}-b^N}{1-b}\sum_{i\in I}U_i(x_j^i)\\
&\quad + \sum_{i\in I}\frac{1-b^N}{1-b^{n_i}}\left[\sum_{j=0}^{n_i-2}\frac{1-b^{j+1}}{1-b}U_i(x_j^i) + \sum_{j=n_i-1}^{N-1}\frac{1-b^{n_i}}{1-b}U_i(x_j^i) + \sum_{j=N}^{N+n_i-2}\frac{b^{j-N+1}-b^{n_i}}{1-b}U_i(x_j^i)\right].
\end{aligned}$$

The two sums on the first line may be combined and factored by the term $(1-b^N)/(1-b)$, which may then be dropped throughout to deduce

$$b\,G_N + \sum_{j=0}^{N-1}\sum_{i\in I}U_i(x_j^i) \geq b\,G_0 + \sum_{i\in I}\left[\sum_{j=0}^{n_i-2}\frac{1-b^{j+1}}{1-b^{n_i}}U_i(x_j^i) + \sum_{j=n_i-1}^{N-1}\frac{1-b^{n_i}}{1-b^{n_i}}U_i(x_j^i) + \sum_{j=N}^{N+n_i-2}\frac{b^{j-N+1}-b^{n_i}}{1-b^{n_i}}U_i(x_j^i)\right].$$

We may now change the order of summation of the first sum, transfer it to the right-hand side (RHS), and cancel out the terms to get

$$b\,G_N \geq b\,G_0 + \sum_{i\in I}\left[\sum_{j=0}^{n_i-2}\frac{b^{n_i}-b^{j+1}}{1-b^{n_i}}U_i(x_j^i) + \sum_{j=N}^{N+n_i-2}\frac{b^{j-N+1}-b^{n_i}}{1-b^{n_i}}U_i(x_j^i)\right],$$

so that dividing by b and rearranging terms, we finally deduce $\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0)$. When $\mathbb{X}_0 \notin \Delta^p$, we have $G_0 > P_0$ and the inequality (12) is strict, so that we get $\Phi(\mathbb{X}_N) > \Phi(\mathbb{X}_0)$. □

Corollary 3.1. *Let $(\mathbb{X}_t)_{t\in\mathbb{N}}$ be any optimal trajectory. Then $\mathbb{X}_t \in \Delta^g \Rightarrow \Phi(\mathbb{X}_{t+N}) \geq \Phi(\mathbb{X}_t)$.*

Proof. Apply the previous result with $\mathbb{X}_t$ as initial state. □

Theorem 3.1 is very general and it holds *under no assumption on the utility functions U_i*. However, it is interesting to notice that since in our setting the U_i's are strictly concave and increasing, then Φ is also strictly concave and attains its maximum at the sustainable state.

Proposition 3.3. *Φ: $\Delta^0 \to \mathbb{R}$ is strictly concave and attains its maximum at $\mathbb{X}^*$.*

Proof. Let us prove first that Φ is strictly concave. The initial terms in the sums defining $G(\mathbb{X}_0)$ are fixed by the initial conditions, so that introducing the function

$$\Psi(\mathbb{X}_0) = \sum_{i\in I}\sum_{t=0}^{n_i-1}\left(\frac{1}{b} - b^t\right)\sigma_i U_i(x_t^i),$$

we may equivalently express Φ as follows:

$$\begin{aligned}\Phi(\mathbb{X}_0) = \Psi(\mathbb{X}_0) + \max\ & \left[\sum_{i\in I}\sum_{t=n_i}^{\infty} b^t U_i(x_t^i)\right]\\ \text{s.t. }\ & \mathbf{x}^i \in l_+^\infty,\\ & \sum_{i\in I} x_{t+n_i}^i = \sum_{i\in I} x_t^i.\end{aligned} \tag{13}$$

The result follows by noting that Ψ is strictly concave on Δ^0, while the maximum is a concave function of $\mathbb{X}_0$, which appears as the RHS of a linearly constrained concave problem.

We prove next that the maximum is attained at $\mathbb{X}^*$. Indeed, for the initial state $\mathbb{X}^*$, the maximum in (13) is attained by $x_t^i = x^{*i}$, which is a stationary point for the Lagrangian (compare with (5))

$$\mathscr{L} = \sum_{i\in I}\sum_{t=n_i}^{\infty}[b^t U_i(x_t^i) + \lambda_t^i x_t^i] + \sum_{t=0}^{\infty}\theta_t\left(\sum_{i\in I} x_t^i - \sum_{i\in I} x_{t+n_i}^i\right) \tag{14}$$

with the following multipliers

$$\begin{cases}\theta_t = b^t r,\\ \lambda_t^i = b^t\left[\dfrac{r}{\sigma_i} - U_i'(x^{*i})\right],\end{cases}$$

where r is the Lagrange multiplier associated with the area constraint in the program (S) that characterizes the sustainable state $\mathbb{X}^*$ (see §2.1). Subdifferential calculus implies that the max function in (13) admits the supergradient $(Y^1, \ldots, Y^k)$ with $Y^i = (\theta_{n_i-1}, \ldots, \theta_0, 0)$, and adding $\nabla\Psi(\mathbb{X}^*)$, we get a supergradient $\mathbb{Y}^* \in \partial\Phi(X^*)$ given by

$$\mathbb{Y}^{*i} = \begin{cases}\dfrac{r}{b}(1,\ldots,1,0) & i \in I^*,\\ \dfrac{\sigma_i}{b}U_i'(0)(1,\ldots,1,0) + [r - \sigma_i U_i'(0)](b^{n_i-1},\ldots,b,1,0) & i \notin I^*.\end{cases}$$

To conclude, we notice that for all $\mathbb{X}_0 \in \Delta^0$, we have

$$\begin{aligned}\langle \mathbb{Y}^*, \mathbb{X}_0 - \mathbb{X}^*\rangle &= \frac{r}{b}\sum_{i\leq i^*}\sum_{t=0}^{n_i-1}(x_t^i - x^{*i}) + \sum_{i>i^*}\sum_{t=0}^{n_i-1}\left[\left(\frac{1}{b} - b^t\right)\sigma_i U_i'(0) + b^t r\right]x_t^i\\ &= \sum_{i>i^*}\sum_{t=0}^{n_i-1}\left(\frac{1}{b} - b^t\right)[\sigma_i U_i'(0) - r]x_t^i \leq 0,\end{aligned}$$

so that $\mathbb{X}^*$ is the unique maximum of Φ over Δ^0. □

3.3. Asymptotic convergence. Since a greedy optimal sequence of states $(\mathbb{X}_t)_{t\in\mathbb{N}}$ is an N-step uphill strategy for Φ, it is natural to expect convergence toward the maximum $\mathbb{X}^*$. We prove that this is the case when $\Delta^p = \{\mathbb{X}^*\}$, while the most one can expect, in general, is convergence to a GPC.

Theorem 3.2. *Let $\mathbb{X}_0 \in \Delta$ be such that the optimal trajectory satisfies $\mathbb{X}_t \in \Delta^g$ for some $t \in \mathbb{N}$. Then, the optimal trajectory converges to a GPC in the sense that*

$$\lim_{t\to\infty} \operatorname{dist}(\mathbb{X}_t, \Delta^p) = 0. \tag{15}$$

In particular, if $\Delta^p = \{\mathbb{X}^*\}$, as in the case $\gcd(n_1, \dots, n_{i^*}) = 1$, *the forest converges to the sustainable state* $\lim_{t\to\infty} \mathbb{X}_t = \mathbb{X}^*$.

Proof. It suffices to establish (15) for which we show that every accumulation point of $\mathbb{X}_t$ belongs to Δ^p. Suppose by contradiction that we have a sequence $t_j \to \infty$ with $\mathbb{X}_{t_j} \to \mathbb{X}^\infty \notin \Delta^p$, and assume with no loss of generality that $\mathbb{X}_{t_j} \in \Delta^g$ for all j. One of the sets $\{i + qN : q \in \mathbb{N}\}$ for $i = 1, \dots, N$ contains infinitely many t_j's, so that passing to a subsequence, we may further assume that $t_j = i + q_j N$ for a fixed i and $q_j \to \infty$.

The set-valued map, which assigns to $\mathbb{X}_0 \in \Delta$ the solution set $S(\mathbb{X}_0)$ of $P(\mathbb{X}_0)$, is upper-semicontinuous with respect to the $\sigma(l^\infty, l^1)$ topology on l^∞. This property combined with the fact that a weak* limit of a greedy trajectory is still greedy, implies that Δ^g is closed, so that $\mathbb{X}^\infty \in \Delta^g$. On the other hand, after Definition 2.2, we observed that for $\mathbb{X}_0 \in \Delta^g$, the optimal solution is unique, so that the map $\mathbb{X}_0 \mapsto S(\mathbb{X}_0)$ is, in fact, strong to weak* continuous from Δ^g to $(l^\infty)^{3k}$. It follows that the map, which assigns to $\mathbb{X}_0 \in \Delta^g$ the state $\mathbb{X}_N \in \Delta^g$ reached at time N is continuous, and then the same holds for the function $\mathbb{X}_0 \in \Delta^g \mapsto \Phi(\mathbb{X}_N) \in \mathbb{R}$.

Now, since $\mathbb{X}^\infty \notin \Delta^p$, Theorem 3.1 gives $\Phi(\mathbb{X}_N^\infty) > \Phi(\mathbb{X}^\infty)$ and by continuity, we may find $\epsilon > 0$ and a neighborhood $\mathcal{V}$ of $\mathbb{X}^\infty$ such that $\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0) + \epsilon$ for all $\mathbb{X}_0 \in \Delta^g \cap \mathcal{V}$. Since $\mathbb{X}_{t_j} \to \mathbb{X}^\infty$, we have $\mathbb{X}_{t_j} \in \Delta^g \cap \mathcal{V}$ for all j large, and then $\Phi(\mathbb{X}_{t_{j+1}}) \geq \Phi(\mathbb{X}_{t_j}) + \epsilon$. This implies $\Phi(\mathbb{X}_{t_j}) \to \infty$, which is impossible since Φ is bounded on Δ^g. This contradiction completes the proof of (15). □

Theorem 3.2 remains true, with the same proof, even if one of the utility functions U_i is merely concave nondecreasing. We will use this observation to extend the result from greedy to general optimal trajectories. The idea is to show that an optimal trajectory for $P(\mathbb{X}_0)$ corresponds to a greedy optimal trajectory in an augmented auxiliary problem, which includes an additional dummy species interpreted as "bare land." In the auxiliary problem, the seedlings are postponed until the time on which they are actually required, transferring the land temporarily to the dummy species in such a way to avoid carrying overmature trees, while keeping the original harvests. More precisely, consider the auxiliary problem

$$\tilde{P}(\mathbb{X}_0) \begin{cases} \text{maximize } \displaystyle\sum_{t=0}^{\infty} b^t \Big[\sum_{i\in I} U_i(u_t^i) + W(w_t) \Big], \\ \text{s.t. } \displaystyle\sum_{i\in I} x_{t+n_i}^i + w_{t+1} = \sum_{i\in I} u_t^i + w_t, \\ \bar{x}_{t+1}^i = \bar{x}_t^i + x_t^i - u_t^i \\ \mathbf{x}^i, \bar{\mathbf{x}}^i, \mathbf{u}^i, \mathbf{w} \in l_+^\infty \end{cases}$$

which adds an annual species with benefit function $W \equiv 0$. Imposing $w = 0$ gives back $P(\mathbb{X}_0)$, so that $\tilde{P}(\mathbb{X}_0)$ is a relaxation and therefore $V(\tilde{P}(\mathbb{X}_0)) \geq V(P(\mathbb{X}_0))$. In fact, both optimal values coincide.

Lemma 3.1. $V(\tilde{P}(\mathbb{X}_0)) = V(P(\mathbb{X}_0))$.

Proof. It suffices to find a feasible trajectory for $P(\mathbb{X}_0)$ with value equal to $V(\tilde{P}(\mathbb{X}_0))$. Let $((\mathbf{y}^i, \bar{\mathbf{y}}^i, \mathbf{u}^i)_{i\in I}, \mathbf{w})$ be an optimal solution for $\tilde{P}(\mathbb{X}_0)$, and let us build $(\mathbf{x}^i, \bar{\mathbf{x}}^i, \mathbf{u}^i)_{i\in I}$ a feasible trajectory for $P(\mathbb{X}_0)$ keeping the harvests unchanged to obtain exactly the same benefit. To this end, we search for $\mu_t^i \geq 0$ such that the following trajectory for $P(\mathbb{X}_0)$ is feasible:

$$\begin{cases} x_t^i = y_t^i + \mu_t^i - \mu_{t-1}^i, \\ \bar{x}_t^i = \bar{y}_t^i + \mu_{t-1}^i, \\ u_t^i = u_t^i. \end{cases} \tag{16}$$

Set $n = \min_{i\in I} n_i$. Arguing as in the proof of Proposition 3.1, it follows that on every interval of length n, there must be at least one $w_t = 0$ (otherwise, we could transfer a small amount of land from w to the species with maturity age n making an extra benefit). Hence we may find an increasing subsequence $\{t_j\}_{j\in\mathbb{N}}$ such that $w_{t_j} = 0$

with $t_0 = 0$ and $|t_{j+1} - t_j| \le n$. Let us define μ^i inductively for $i = 1, 2, \ldots, k$ setting $\mu^i_t = 0$ for $t < n_i$, while for every $j \ge 1$, we put

$$\begin{cases} \mu^i_{t_j+n_i-1} = 0, \\ \mu^i_{t+n_i-1} = \min\left(w_t - \sum_{l=1}^{i-1} \mu^l_{t+n_l-1}, y^i_{t+n_i} + \mu^i_{t+n_i}\right) & t_{j-1} < t < t_j, \end{cases}$$

where the last recursion is solved backward starting from $t = t_j - 1$ down to $t = t_{j-1} + 1$.

Let us show that the trajectory (16) is feasible. The inequality $\mu^i_{t+n_i-1} \le w_t - \sum_{l=1}^{i-1} \mu^l_{t+n_l-1}$ inductively implies $\mu^i \ge 0$, so that $\bar{x}^i \ge 0$, while $x^i \ge 0$ comes from the fact that $\mu^i_{t-1} \le y^i_t + \mu^i_t$. Since the equality $\bar{x}^i_{t+1} = \bar{x}^i_t + x^i_t - u^i_t$ is checked by direct substitution, it remains to prove $\sum_{i\in I} u^i_t = \sum_{i\in I} x^i_{t+n_i}$, which will follow from the area balance of $\tilde{P}(\mathbb{X}_0)$ if we show that $\sum_{i\in I} x^i_{t+n_i} = \sum_{i\in I} y^i_{t+n_i} + w_t$. To this end, it suffices to prove that

$$w_t = \sum_{i\in I} \mu^i_{t+n_i-1}. \tag{17}$$

We do so by backward induction. This obviously holds for $t = t_j$ since $w_{t_j} = 0$ and $\mu^i_{t_j+n_i-1} = 0$ for all $i \in I$. Suppose that (17) holds at time $t+1$ and let us show that it still holds at time t. We claim that there exists some i such that the minimum in the definition of $\mu^i_{t+n_i-1}$ is attained in the first term, since otherwise we would have

$$w_t > \sum_{i\in I} \mu^i_{t+n_i-1} = \sum_{i\in I} (y^i_{t+n_i} + \mu^i_{t+n_i}) = \sum_{i\in I} y^i_{t+n_i} + w_{t+1},$$

which contradicts the area balance in $\tilde{P}(\mathbb{X}_0)$. We deduce $\mu^i_{t+n_i-1} = w_t - \sum_{l=1}^{i-1} \mu^l_{t+n_l-1}$ and then $\mu^l_{t+n_l-1} = 0$ for all $l > i$, which yields $w_t = \sum_{l\in I} \mu^l_{t+n_l-1}$ completing the induction step. □

We may now state our main result on the asymptotic behavior of an optimally managed forest. This extends Theorem 3.1 by removing the assumption $\mathbb{X}_0 \in \Delta^g$, henceforth showing that all optimal trajectories converge toward the set of GPCs.

THEOREM 3.3. *Every optimal trajectory of $P(\mathbb{X}_0)$ converges to a GPC in the sense that*

$$\lim_{t\to\infty} \operatorname{dist}(\mathbb{X}_t, \Delta^p) = 0. \tag{18}$$

In particular, if $\Delta^p = \{\mathbb{X}^*\}$, as in the case $\gcd(n_1, \ldots, n_{i^*}) = 1$, *the forest converges to the sustainable state* $\lim_{t\to\infty} \mathbb{X}_t = \mathbb{X}^*$.

PROOF. Let $(\mathbf{x}^i, \bar{\mathbf{x}}^i, \mathbf{u}^i)_{i\in I}$ be an optimal solution for $P(\mathbb{X}_0)$ and consider an equivalent greedy optimal trajectory $((\mathbf{y}^i, \bar{\mathbf{y}}^i, \mathbf{u}^i)_{i\in I}, \mathbf{w})$ for $\tilde{P}(\mathbb{X}_0)$. We know that $\bar{x}^i_t = 0$ for at least one $t \in [n_i, 2n_i - 1]$. Let t_i be any such t and define $w_t = \sum_{i\in I} \bar{x}^i_{t+n_i} \mathbb{1}_{\{t+n_i \ge t_i\}}$ and

$$\begin{cases} y^i_t = x^i_t;\ \bar{y}^i_t = \bar{x}^i_t & t < t_i, \\ y^i_t = u^i_t;\ \bar{y}^i_t = 0 & t \ge t_i. \end{cases}$$

This trajectory is feasible for $\tilde{P}(\mathbb{X}_0)$ and becomes greedy for $t \ge \max_{i\in I} t_i$. As we have kept the same harvests u^i_t, the previous lemma implies that the trajectory is also optimal for $\tilde{P}(\mathbb{X}_0)$.

Although the function $W \equiv 0$ is only concave and not strictly concave, the auxiliary problem has still a unique sustainable state in which the dummy species w is not present (Problem (S) has a unique solution even if one of the utility functions is merely concave). Moreover, it is easy to see the set of states leading to GPCs in the auxiliary problem is given by $\tilde{\Delta}^p = \Delta^p \times \{0\}$. Using the remark after Theorem 3.2, we deduce that the modified greedy optimal trajectory converges to $\Delta^p \times \{0\}$. It follows that for the auxiliary optimal path, we have $\lim_{t\to\infty} w_t = 0$, so that $\bar{x}^i_t \to 0$ and then $|x^i_t - y^i_t| \to 0$, which allows to conclude (18). □

As a consequence of this result, we deduce that every optimal trajectory is *asymptotically* greedy. We already mentioned that for the one species case, optimal trajectories become greedy after a finite time, while §3.1 described situations in which this occurs for multiple species. In general, however, it is an open question whether optimal trajectories become greedy or not.

When $\Delta^p = \{\mathbb{X}^*\}$, the previous result implies a turnpike property for the optimal harvesting. Such results are known for wide classes of dynamic optimization problems, though most of them assume discount factors close to one, while smaller factors may induce complex behaviors, including chaos (cf. Boldrin and Montrucchio [1], Le Van et al. [3], Majumdar et al. [4], McKenzie [5, 6], Mitra and Nishimura [7], Montrucchio [9], Zaslavski [17]). In contrast, our results hold for all values of $b \in (0, 1)$ provided that $\Delta^p = \{\mathbb{X}^*\}$.

4. Finite-time convergence. We have seen that an optimal trajectory approaches a GPC or even the sustainable state. Thus, it is worth investigating if this convergence occurs in finite time, in which case $P(\mathbb{X}_0)$ could be reformulated as a finite-dimensional problem and solved numerically. Interestingly, this is not always the case even for states in Δ^g, which admit a greedy optimal trajectory but for which the dynamics might take an infinite time to approach asymptotically a *periodic* trajectory. We restrict the analysis to a case with two species because the general case is too involved. In the sequel, we denote $t(n)$ the integer t modulo n.

Proposition 4.1. *Suppose* $\gcd(n_1, n_2) = 1$ *and* $I^* = \{1, 2\}$. *For each* $\mathbb{X}_0 \in \Delta^g$, *either the optimal trajectory reaches* $\mathbb{X}^*$ *within* $n = \min\{n_1, n_2\}$ *steps or the convergence is asymptotic.*

Proof. The case $n_1 = n_2 = 1$ being trivial, we just consider the case $n_1 \neq n_2$. Assume with no loss of generality that $n_1 > n_2$ and suppose that the sustainable state is reached at time T. If $T > n_2$, we consider a perturbed trajectory obtained by moving a small area from X^1 to X^2 in the sowing at stage $T - n_2 - 1$

$$\tilde{X}_t^1 = \begin{cases} x_t^1 - \epsilon & \text{if } t = T - n_2 - 1 + i n_1, \quad i \geq 1, \\ x_t^1 & \text{otherwise.} \end{cases} \qquad \tilde{X}_t^2 = \begin{cases} x_t^2 + \epsilon & \text{if } t = T - 1 + i n_2, \quad i \geq 0, \\ x_t^2 & \text{otherwise.} \end{cases}$$

Observing that $x_t^1 = x^{*1} > 0$ and $x_t^2 = x^{*2}$ for all $t \geq T$, we conclude that the new trajectory is feasible. Thus, its benefit must be smaller than the optimum

$$b^{T-n_2-1}\sigma_1[U_1(x^{*1} - \epsilon) - U_1(x^{*1})] + b^{T-1}[U_2(x_{T-1}^2 + \epsilon) - U_2(x_{T-1}^2)] + b^{T-1}\sigma_2[U_2(x^{*2} + \epsilon) - U_2(x^{*2})] \leq 0.$$

Dividing by $\epsilon > 0$ and letting it to 0, we get $U_2'(x^{*2}) \geq U_2'(x_{T-1}^2)$, which combined with the strict concavity of U_2, implies $x_{T-1}^2 \geq x^{*2} > 0$. The previous inequality also holds for $\epsilon < 0$ small, so that dividing again by $\epsilon < 0$ and letting it to 0, we deduce $U_2'(x_{T-1}^2) = U_2'(x^{*2})$. Hence, $x_{T-1}^2 = x^{*2}$ and a balance of area at stage $T - 1$ yields $x_{T-1}^1 = x^{*1}$, showing that the sustainable state was reached, in fact, at $T - 1$. It follows by backward induction that the sustainable state is reached at a time $T \leq n_2$. □

We may explicitly characterize the initial conditions for which the corresponding optimal trajectories attain the sustainable state in finite time.

Proposition 4.2. *Suppose* $n_1 \geq n_2$, $\gcd(n_1, n_2) = 1$ *and* $I^* = \{1, 2\}$. *For a given* $\mathbb{X}_0 \in \Delta^g$, *the optimal trajectory reaches the sustainable* $\mathbb{X}^*$ *state within* n_2 *steps if and only if*
(a) $x_t^1 + x_t^2 = x^{*1} + x^{*2}$ *for* $t = 0, \ldots, n_2 - 1$,
(b) $x_t^1 = x^{*1}$ *for* $t = n_2, \ldots, n_1 - 1$.

Proof. Conditions (a) and (b) are clearly necessary to reach the sustainable state $\mathbb{X}^*$ within n_2 steps. Conversely, we must show that the trajectory $x_t^1 = x^{*1}$ for $t \geq n_1$ and $x_t^2 = x^{*2}$ for $t \geq n_2$ is optimal. To this end, we consider the Lagrangian (14) and the multipliers $\lambda_t^1 = \lambda_t^2 = 0$ and $\theta_t = b^t r$. It is easy to see that $\nabla L = 0$ and that we have complementary slackness so the proposed trajectory is optimal. □

We analyze next the finite convergence to Δ^p when $I^* = \{1\}$ by considering separately the cases $n_1 > n_2$ and $n_1 < n_2$.

Proposition 4.3. *Suppose* $n_1 > n_2$ *and* $I^* = \{1\}$. *For each* $\mathbb{X}_0 \in \Delta^g$, *either the optimal trajectory reaches* Δ^p *within* n_2 *time steps or the convergence is asymptotic.*

Proof. Let us assume that the optimal trajectory reaches a state $\widehat{\mathbb{X}} \in \Delta^p$ at time $T > n_2$. If $x_{T-1}^2 > 0$, we consider the perturbed trajectory obtained by moving a small area from species X^2 to X^1 in the sowing at stage $T - n_2 - 1$, which is then returned to X^2 at time $T - 1$

$$\tilde{X}_t^1 = \begin{cases} x_t^1 + \epsilon & \text{if } t = T - n_2 - 1 + i n_1, \quad i \geq 1, \\ x_t^1 - \epsilon & \text{if } t = T - 1 + i n_1, \quad i \geq 1, \\ x_t^1 & \text{otherwise.} \end{cases} \qquad \tilde{X}_t^2 = \begin{cases} x_t^2 - \epsilon & \text{if } t = T - 1, \\ x_t^2 & \text{otherwise.} \end{cases}$$

To see that the proposed alternative trajectory is feasible for some $0 < \epsilon < x_{T-1}^2$, we need only to check that $x_{T-1+in_i}^1 - \epsilon > 0$, which follows directly from the area balance $x_{T-1}^1 + x_{T-1}^2 = x_{T-1+n_1}^1 + x_{T-1+n_2}^2$ together with $x_{T-1+n_2}^2 = 0$ and the periodicity of x_t^i beyond T. As the alternative trajectory is feasible, its benefit must be smaller than the optimum. Given that for $t \geq T$ we have reached the GPC, a straightforward computation yields

$$[U_2(x_{T-1}^2 - \epsilon) - U_2(x_{T-1}^2)] + b^{-n_2}\sigma_1[U_1(\hat{x}_{n_1-n_2-1}^1 + \epsilon) - U_1(\hat{x}_{n_1-n_2-1}^1)] + \sigma_1[U_1(\hat{x}_{n_1-1}^1 - \epsilon) - U_1(\hat{x}_{n_1-1}^1)] \leq 0.$$

Dividing by ϵ and letting it to 0, we get

$$U_2'(x^2_{T-1}) \geq \sigma_1\left[\frac{1}{b^{n_2}}U_1'(\hat{x}^1_{n_1-n_2-1}) - U_1'(\hat{x}^1_{n_1-1})\right]. \tag{19}$$

If $\hat{x}^1_{n_1-n_2-1}=0$, (19) leads to a contradiction: $U_2'(x^2_{T-1}) \geq \sigma_1(1/b^{n_2}-1)U_1'(0) \geq U_2'(0)$. The same contradiction can be obtained when $\hat{x}^1_{n_1-n_2-1}>0$ combining (19) with (7b). We thus get $x^2_{T-1}=0$ and then the area balance at stage $T-1$ yields $x^1_{T-1}=\hat{x}^1_{n-1}$, which implies $\mathbb{X}_{T-1}\in\Delta^p$, showing that Δ^p was reached, in fact, at stage $T-1$. Proceeding inductively, we conclude that convergence must occur before stage n_2. □

Again, it is of interest to characterize the set of initial states whose optimal trajectories reach the set Δ^p in at most n_2 steps.

Proposition 4.4. *Suppose $n_1>n_2$ and $I^*=\{1\}$. For a given $\mathbb{X}_0\in\Delta^g$, the optimal trajectory reaches Δ^p within n_2 steps if and only if $\widehat{\mathbb{X}}_0\in\Delta^p$, where*

$$\begin{cases} \widehat{X}^1_0=(x^1_{n_1-1},\ldots,x^1_{n_2},x^1_{n_2-1}+x^2_{n_2-1},\ldots,x^1_0+x^2_0,0),\\ \widehat{X}^2_0=(0,\ldots,0). \end{cases}$$

Proof. It is clear that $\widehat{\mathbb{X}}_0\in\Delta^p$ is necessary to attain Δ^p within n_2 steps. Conversely, we must show that the trajectory $x^1_t=\hat{x}^1_{t(n_1)}$ for $t\geq n_1$ and $x^2_t=0$ for $t\geq n_2$ is optimal, for which it suffices to observe that this trajectory is a stationary point for the Lagrangian (14) with respect to the following multipliers:

$$\begin{cases} \theta_t=b^t\sigma_1U_1'(x^1_t),\\ \lambda^i_t=\theta_{t-n_i}-\theta_t-b^tU_i'(x^i_t). \end{cases}$$

It is easy to see that $\lambda^1_t=0$ for all $t\geq n_1$. And $\lambda^2_t\geq 0$ for all $t\geq n_2$ follows directly from (7b) whenever $\hat{x}^1_{t-n_2}>0$ and from $\sigma_1U_1'(0)\geq\sigma_2U_2'(0)$ and the concavity of U_i when $\hat{x}^1_{t-n_2}=0$ □

The results for the case $n_1<n_2$ are similar but the proofs are more technical.

Proposition 4.5. *Suppose $n_1<n_2$ and $I^*=\{1\}$. For $\mathbb{X}_0\in\Delta^g$, either the optimal trajectory reaches Δ^p within n_2 steps or the convergence is asymptotic.*

Proof. As in the previous proofs, assume that the optimal trajectory reaches a state $\widehat{\mathbb{X}}\in\Delta^p$ at time $T>n_2$. If $x^2_{T-1}>0$, we backtrack to stage $T-n_2-1$ when x^2_{T-1} was sown and perturb the optimal trajectory by sowing $x^2_{T-1}-\epsilon$ and transferring this ϵ to species 1. At stage $T+n_1-n_2-1$, we return the extra ϵ to X^2, whereas at $T-1$, we assign the deficit to X^1 to continue with the original trajectory after $T+n_1-1$. Namely, we are considering the following alternative trajectory:

$$\widetilde{X}^1_t=\begin{cases} x^1_t+\epsilon & \text{if } t=T+n_1-n_2-1,\\ x^1_t-\epsilon & \text{if } t=T+n_1-1,\\ x^1_t & \text{otherwise;}\end{cases}\qquad \widetilde{X}^2_t=\begin{cases} x^2_t-\epsilon & \text{if } t=T-1,\\ x^2_t+\epsilon & \text{if } t=T+n_1-1,\\ x^2_t & \text{otherwise,}\end{cases}$$

which is feasible for $\epsilon>0$ small, because the area balance at stage $T-1$ implies $x^1_{T+n_1-1}>0$. The benefit of the new trajectory must be smaller than the optimum. Since at time T, we reach the state $\widehat{\mathbb{X}}$, we deduce

$$\begin{aligned} &b^{n_1-n_2}[U_1(x^1_{T+n_1-n_2-1}+\epsilon)-U_1(x^1_{T+n_1-n_2-1})]+[U_2(x^2_{T-1}-\epsilon)-U_2(x^2_{T-1})]\\ &\quad+b^{n_1}[U_1(\hat{x}^1_{n_1-1}-\epsilon)-U_1(\hat{x}^1_{n_1-1})+U_2(\epsilon)-U_2(0)]\leq 0,\end{aligned}$$

so that dividing as usual by ϵ and letting it to 0, we obtain

$$b^{n_1-n_2}U_1'(x^1_{T+n_1-n_2-1})-U_2'(x^2_{T-1})+b^{n_1}[-U_1'(\hat{x}^1_{n_1-1})+U_2'(0)]\leq 0.$$

Since $x^2_t=0$ for all $t\geq T$, repeated application of the area balance allows us to write $x^1_{T+n_1-n_2-1}\leq\hat{x}^1_{n_1-n_2-1(n_1)}$ and using the fact that $x^2_{T-1}>0$, the last inequality yields

$$U_2'(0)>\sigma_1\left[\frac{1}{b^{n_2}}U_1'(\hat{x}^1_{n_1-n_2-1(n_1)})-U_1'(\hat{x}^1_{n_1-1})\right],$$

contradicting $\widehat{\mathbb{X}}\in\Delta^p$ (consider condition (7b) of Theorem 2.2). This contradiction implies $x^2_{T-1}=0$ and then the area balance at stage $T-1$ gives $x^1_{T-1}=x^1_{T+n_1-1}=\hat{x}^1_{n_1-1}$, so that $\mathbb{X}_{T-1}\in\Delta^p$. We conclude inductively that Δ^p is reached within n_2 steps. □

PROPOSITION 4.6. *Suppose $n_1 < n_2$ and $I^* = \{1\}$. For a given $\mathbb{X}_0 \in \Delta^g$, the optimal trajectory reaches Δ^p within n_2 steps if and only if $\widehat{\mathbb{X}}_0 \in \Delta^p$, where $\widehat{X}_0^2 = 0$ and $\hat{x}_t^1 = x_t^1 + \sum_{i=0}^{n_2-1} x_i^2 \mathbb{1}_{\{i=t(n_1)\}}$ for all $t = 0, \ldots, n_1 - 1$.*

PROOF. Obviously, Δ^p may not be reached with a greedy policy in n_2 stages if $\widehat{\mathbb{X}}_0 \notin \Delta^p$. Conversely, let us show that the trajectory $x_t^2 = 0$ for $t \geq n_2$ and

$$x_t^1 = \begin{cases} x_{t(n_1)}^1 + \displaystyle\sum_{i=0}^{t-1} x_i^2 \mathbb{1}_{\{i=t(n_1)\}} & t = n_1, \ldots, n_2 - 1, \\ \hat{x}_{t(n_1)}^1 & t \geq n_2 \end{cases}$$

is optimal. We consider the Lagrangian in (14) and the following l^1-multipliers:

$$\begin{cases} \theta_t = b^t \sigma_1 U_1'(x_t^1) & t \geq n_2; \\ \theta_t = b^t \left[\displaystyle\sum_{j=1}^{l} b^{jn_1} U_1'(x_{t+jn_1}^1) \right] + \theta_{t+ln_1} & t < n_2, \quad \text{where } t + ln_1 \in [n_2, n_1 + n_2); \\ \lambda_t^i = \theta_{t-n_i} - \theta_t - b^t U_i'(x_t^i) & t \geq n_i. \end{cases}$$

A cumbersome computation shows that $\nabla L = 0$ and that we have complementary slackness, so the proposed trajectory is optimal. □

5. Conclusion. In this paper, we discussed a model for the optimal management of a mixed forest composed by several species with different maturity ages, under the restriction that only mature trees can be harvested. The model can serve as a prototype for the exploitation of a finite resource such as land or space, which can be allocated to different activities that produce their revenue after certain delays at which the resource is liberated for reuse.

We established the existence and uniqueness of a *sustainable state*, which is invariant under the optimal harvesting policy, and which was characterized as the unique solution of a finite-dimensional optimization problem. We also discussed the existence of periodic optimal solutions and showed that if $\gcd(n_1, \ldots, n_{i^*}) = 1$, then the only periodic solution is the sustainable state.

In our main result, we proved that any optimally managed forest must converge toward the set of GPCs. In particular, when $\gcd(n_1, \ldots, n_{i^*}) = 1$, all optimal trajectories converge to the sustainable state. The key for the asymptotic analysis was to identify a pseudo-Lyapunov function, which increases every N steps, where N is the least common multiple of the maturity ages $n_1, \ldots, n_k$ of the species considered.

As a by-product of the convergence analysis, we obtained that in the long run, any optimal trajectory becomes asymptotically greedy in the sense that the areas covered by overmature trees converge to zero. We also described some particular cases in which the greedy regime is attained after a *finite*-initial phase in which over-mature areas are required for transferring area between different age classes and species. However, the finite-time convergence to a greedy regime remains an open question in the general case. On the other hand, we showed that the convergence toward a GPC may, in fact, be only asymptotic and require an infinite time, even if we start from an initial state whose corresponding optimal solution is known to be greedy. More precisely, for a two-species forest, we characterized the set of initial conditions for which this finite-time convergence holds. This is a significant difference with respect to a one-species forest, which always converges in finite time to an optimal GPC.

Another open problem is that, although we established the convergence of $\mathbb{X}_t$ toward the set Δ^p, we did not prove the convergence to a particular GPC, except, of course, when $\Delta^p = \{\mathbb{X}^*\}$. A positive answer to this question and the characterization of the basins of attraction would be of interest. In the simple case of a two-species forest with maturity ages $n_1 = 2$ and $n_2 = 1$, we could prove convergence to a particular GPC, which depends on the initial condition, but the analysis is too specific and does not extend to higher dimensions, so we did not present it.

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