Economics Letters 109 (2010) 94–98

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

On the non-existence of optimal programs in the Robinson–Solow–Srinivasan (RSS) model

M. Ali Khan [a,*], Adriana Piazza [b]

[a] Department of Economics, The Johns Hopkins University, Baltimore, MD 21218 and Department of Economics, Nanyang Technological University, 14 Nanyang Drive, 637332 Singapore
[b] Departamento de Matemática, Universidad Técnica Federico Santa María. Avda. España 1680, Casilla 110-V, Valparaíso, Chile

ARTICLE INFO

Article history:
Received 5 September 2009
Received in revised form 29 July 2010
Accepted 5 August 2010
Available online 30 September 2010

JEL classification:
C62
D90
Q23

Keywords:
RSS model
Optimal programs
Brock prices
Minimum value-loss programs
Césaro means convergence

ABSTRACT

We show that in the 2-sector RSS model, there is no optimal program for any initial stock when the felicity function is linear and the marginal rate ξ equals unity. This settles a conjecture, unresolved since 2005.

1. Introduction

Since the revisiting by Khan and Mitra (2005a) of a model due to Robinson (1960), Solow (1962) and Srinivasan (1962) dating to the early sixties,[1] and with further analysis by Stiglitz (1968, 1973) and Cass and Stiglitz (1970) in the early seventies, the results have been fast and free-flowing.[2] To be sure, the so-called RSS model, perhaps not unlike its smoother one-sector cousin, the so-called RCK model,[3] is hardly of interest for its mirroring of the complexities of any modern economy, but rather because it yields several novel methodological principles. In particular, in calling upon the general theory of intertemporal allocation as developed by Gale (1967) and McKenzie (1986, 2002), it draws attention to those parts of the theory that most need development. In addition to questions of transition dynamics and their dependence on the discount factor, nothing if not curious,[4] the most dramatic of these is the total dissonance between the continuous and discrete-time analysis, an indeterminacy evident in the non-uniqueness of *maximal* programs starting from the same initial stock, and a "folk-theorem" which identifies a threshold discount factor above which the undiscounted and discounted dynamics are identical.[5] In short, the RSS technological specification, and ironically its 2-sector specialization, is one that seems to have fallen in the cracks between the 1-sector RCK technology and that of the 2-sector neoclassical model due to Shinkai, Uzawa and Srinivasan.[6]

In drawing on the framework and theorems of the Gale-McKenzie (GM) reduced-form model, the modern RSS analysis, as opposed to the earlier one of Cass and Stiglitz,[7] illustrates rather well the tension implicit in the subsequent literature on the GM model. This pertains to alternative criteria for the undiscounted setting that coalesce in the discounted one. In his pioneering study, Gale showed the existence of *optimal* programs under the assumption of strictly

* Corresponding author.
E-mail addresses: akhan@jhu.edu (M.A. Khan), adriana.piazza@usm.cl (A. Piazza).

[1] Also see Mirrlees (1962) and Okishio (1966).

[2] See Khan and Mitra (2005,2006,2007a,b,2008b), Khan and Zaslavski (2007,2008,2010a,b), Zaslavski (2005, 2007, 2010) and their references.

[3] For the RCK model due to Ramsey, Cass and Koopmans, with its smooth neoclassical technology, the reader is referred to any modern textbook in macroeconomics.

[4] See the results on this dependence in Khan and Mitra (2008a); also Metcalf (2008).

[5] To the best of the authors' knowledge, the first articulation of this "folk-theorem" that we could find in the literature is in Khan (2005). The idea received further elaboration in Khan and Mitra (2007b) (see the Introduction), and its primary impulse is rooted in McKenzie's neighborhood turnpike theorem; see his 1982, 1983 JET papers referred to in McKenzie (1986, 2002).

[6] See Fujio (2009), Nishimura and Yano (1995) and their references to the work of Shinkai, Uzawa, Srinivasan and others on the 2-sector model.

[7] This essentially means working in the space of stocks rather than with both state and control variables as is the case in Pontryagin's maximum principle.

0165-1765/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.econlet.2010.08.010

concave felicities, but as Brock (1970) emphasized soon after, one cannot expect optimal programs to exist in the case of linear felicities, and that one had to work with a weakened criterion of *maximal* programs to get an analytically viable maximand and an existence proof. Both Brock's counterexample, and his theorem with a method of proof relying on Césaro means convergence, Cantor's diagonalization argument, and what is referred to as Brock prices by Fujio and Khan (2007), and Peleg's (1973) subsequent strengthening of the example, have all proved influential.[8] Given the importance of linear felicities,[9] subsequent work has been phrased in terms of the maximality criterion; and in terms of the Atsumi (1965) and von Weizsäcker (1965) origins, in working with a program that merely *catches up* to another, as opposed to one that cannot be *overtaken* by any other. However, in recent work by Zaslavski (2005), and following him, by Khan and Zaslavski (2007, 2008), there has been a substantial rehabilitation of the optimality criterion even when, in Brock's words, the "objectionable hypothesis of strict concavity of the utility function" does not hold. In particular, it has been shown by Zaslavski (2005) that there exists an optimal program when the rate of transformation ξ_σ between the (unique) golden-rule type σ of machines today and machines tomorrow, and with zero consumption, is not unity. For the case when this rate is unity, it was shown by Khan and Mitra (2006, 2007) that maximal programs are not necessarily minimal value-loss programs, and thereby not necessarily unique. Zaslavski (2005) builds on these results to present a maximal program that is not an optimal program, and states:

> "It should be mentioned that in the case $\xi_\sigma = 1$, Khan and Mitra (2006) established non-uniqueness of maximal programs starting from the same initial stock. Probably in this case an optimal program does not exist."

The question of the non-existence of an optimal program in the RSS model has since then remained open and proved elusive. In this paper we give a decisively negative answer to the question.

In Section 2, we present the RSS model and its basic vocabulary, and in Section 3, a result showing that every optimal program is necessarily a minimal value-loss program. Section 4 presents the example in terms of the 2-sector RSS model, and Section 5 delineates two questions opened by our results.

2. The model and basic concepts

We begin with a brief description of the model and previous results necessary for this paper. The reader is referred to Khan and Mitra (2005a), Zaslavski (2007), Khan and Zaslavski (2010b) for more details.

For this and the next section, we shall be working in the non-negative orthant of an n-dimensional Euclidean space $\mathbb{R}^n_+$, and the set of non-negative integers $\mathbb{N}$. The *transition possibility set*

$$\Omega = \left\{(x,x') \in \mathbb{R}^n_+ : x' - (1-d)x \geq 0 \text{ and } a(x' - (1-d)x) \leq 1\right\},$$

where x represents the stock of machines today, x' the stock tomorrow, d the rate of depreciation lying in $(0,1)$, and the strictly positive a_i the labor coefficients required to manufacture a machine of type i, $i = 1,\ldots,n$. Given the pair $(x,x') \in \Omega$ and denoting by e the sum vector of $\mathbb{R}^n$, the stock of machines devoted to the consumption goods sector is given by the correspondence

$$\Lambda(x,x') = \left\{y \in \mathbb{R}^n_+ : 0 \leq y \leq x \text{ and } ey \geq 1 - a(x' - (1-d)x)\right\}.$$

For every $i = 1,\ldots,n$, one unit of machine together with one unit of labor, can produce $b_i > 0$ units of a single consumption good, thus, defining the output-coefficients vector $b = (b_1,\ldots,b_n) \in \mathbb{R}^n_+$, the total good production is given by by.

Let $w: \mathbb{R}_+ \to \mathbb{R}$ be a continuous, strictly increasing concave differentiable function representing the planner's preferences. Once the initial stock of machines x_0 is given, the specification of the RSS model is complete.

We shall work with the following concepts.

Definition 2.1. A program from x_0 is a sequence $\{x(t),y(t)\}$ such that $x(0) = x_0$ and for all $t \in \mathbb{N}$, $((x(t),x(t+1)) \in \Omega$ and $y(t) \in \Lambda(x(t), x(t+1))$. A program $\{x^*(t),y^*(t)\}$ is optimal if for any program $\{x(t), y(t)\}$ such that $x(0) = x^*(0)$ we have

$$\limsup_{T\to\infty} \sum_{t=0}^{T} w(by(t)) - w\left(by^*(t)\right) \leq 0.$$

A program is maximal if the lim sup operator above is substituted for lim *inf*.

The pair $(\hat{x}, \hat{p})$ is a basic benchmark of the model.

Definition 2.2. The golden-rule stock (GRS), $\hat{x}$, is the solution to the problem $\max\{u(x,x') : (x,x') \in \Omega \text{ and } x \leq x'\}$, where $u(x,x') = \max\{w(by) : y \in \Lambda(x,x'), (x,x') \in \Omega\}$, $b \in \mathbb{R}^n_+$. The golden-rule stock price, $\hat{p} \in \mathbb{R}^n_+ / 0$, is the solution to

$$u(\hat{x},\hat{x}) \geq u(x,x') - \hat{p}(x' - x) \text{ for all } (x,x') \in \Omega. \tag{1}$$

We can now present our final set of concepts. First, the function $\delta : \Omega \to \mathbb{R}_+$ which follows directly from Eq. (1), and second two different kinds of programs. The first due to Gale (1967) and the second to Brock (1970).

Definition 2.3. The value-loss associated with any $(x,x') \in \Omega$ is given by

$$\delta(x,x') = u(\hat{x},\hat{x}) - u(x,x') - \hat{p}(x' - x) \geq 0.$$

Definition 2.4. A program $\{x(t),y(t)\}$ is good if there exists $G \in \mathbb{R}$ such that $\sum_{t=0}^{T}[w(by(t)) - w(b\hat{y})] \geq G$ for all $T \in \mathbb{N}$, where $\hat{y} = \arg\max\{w(by) : y \in \Lambda(\hat{x},\hat{x})\} = \hat{x}$. A program $\{x(t),y(t)\}$ is a minimal value-loss program from x_0 if it attains the infimum of $\sum_{t=0}^{\infty}\delta(x(t), x(t+1))$ over all programs from x_0.

So far, we have been silent on the question as to whether the concepts are well-defined. Brock (1970) showed that minimal value-loss programs exist in the generality of the GM model, and under the assumption of a unique *GRS*, made the insightful observation that the Cesáro means of every good program $\{x(t),y(t)\}$ converge to $\hat{x}$; namely,

$$\bar{x}(t) = \frac{1}{t}\sum_{l=0}^{t-1} x(l) \to \hat{x} \quad \text{when } t \to \infty.$$

Since a minimal value-loss program is a good program, he then relied on the convergence property to show that every minimal value-loss program is a maximal program.

Turning to the special case of the RSS model, for the case of a single technique, $n = 1$, there is an example of a maximal program that is not a minimal value-loss program; see Khan and Mitra (2006, 2007). It is also shown that maximal programs are not necessarily unique. The fact that every optimal program is maximal is a

[8] Brock prices are introduced to avoid "using the laborious method of competitive prices" of Gale's. Brock also refers to our *maximal* notion as *weakly maximal*. We are grateful to Tapan Mitra for bringing (Peleg, 1973) to our attention.

[9] This is so not only in work on the RSS model and its 2-sector variant, but also on the Mitra-Wan (MW) tree farm, see Khan and Piazza (2009) and the references to the papers of Mitra and of Mitra-Wan.

triviality, but according to Zaslavski (2005), there is an example of a maximal program that is not optimal. In short, all this work highlights the fact that Brock's theorem establishes the *sufficiency* of value-loss minimization for maximality, and that the *necessity* is false.

3. A simple result

All of the negative results mentioned in the conclusion to Section 2 above relate to a particular configuration of parameters in the two-sector RSS model; namely when the marginal rate of transformation $\xi=(1/a)-(1-d)$ is unity. For the general case where this does not hold, which is to say $\xi_\sigma\neq 1$, σ being the unique golden-rule machine-type given by $\arg\max_i(b_i/(1+a_i d))$, the notions of optimal, maximal and minimal value-loss programs are all identical. As shown in (Zaslavski (2007), Theorem 2.4), it is the asymptotic convergence of the stocks of any good program to the *GRS* that turns out to be a key ingredient in the demonstration that any minimal value-loss program is optimal, and hence for the proof of the existence of such programs.

What seems to have been missed out is the fact that every optimal program is a minimal value-loss program, and ironically, it is the convergence of the Cesáro means that proves basic to its demonstration.

Theorem 3.1. *If $\{x^*(t),y^*(t)\}$ is an optimal program from x_0, then it is a minimal value-loss program.*

Proof. Suppose, contrary to our claim, that the program $\{x^*(t),y^*(t)\}$ does not minimize the accumulated value-loss and let $\{x(t),y(t)\}$ be a minimizer. Hence, there exists $\epsilon>0$ such that

$$\sum_{t=0}^{\infty}\delta(x^*(t),x^*(t+1))-\sum_{t=0}^{\infty}\delta(x(t),x(t+1))>\epsilon.$$

And given $\epsilon_0\in(0,\epsilon)$, there is T_0 such that

$$\sum_{t=0}^{T-1}\delta(x^*(t),x^*(t+1))-\sum_{t=0}^{T-1}\delta(x(t),x(t+1))>\epsilon_0 \quad \text{for all } T\geq T_0.$$

Using the very easy to check equality

$$\begin{aligned}\sum_{t=0}^{T-1}\left[w(by(t))-w\left(by\ (t)\right)\right] &= \sum_{t=0}^{T-1}\delta(x^*(t),x^*(t+1))\\ &\quad -\sum_{t=0}^{T-1}\delta(x(t),x(t+1))\\ &\quad +\hat{p}\ (x^*(T)-x(T)),\end{aligned}$$

we deduce

$$\sum_{t=0}^{T-1}[w(by(t))-w(by^*(t))]>\epsilon_0+\hat{p}(x^*(T)-x(T))\text{for all } T\geq T_0.$$

We know as well that there is T_1 such that

$$\limsup_{T\to\infty}\sum_{t=0}^{T-1}[w(by(t))-w(by^*(t))]+\epsilon_0/2>\sum_{t=0}^{T-1}[w(by(t))-w(by^*(t))] \quad \text{for all } T\geq T_1.$$

Using the last two inequalities and the optimality of $\{x^*(t),y^*(t)\}$ we get

$$\epsilon_0/2>\epsilon_0+\hat{p}(x^*(T)-x(T)) \quad \text{for all } T\geq max\{T_0,T_1\}$$

which leads to the following contradiction $-\varepsilon_0/2\geq lim_{T\to\infty}\hat{p}\left(x\ (T)-\ x(T)\right)=0.$ □

4. The example

We now turn to the question of the possible non-existence of an optimal program. Towards this end, we work with the 2-sector model. When $n=1,\xi=(1/a)-(1-d)=1$, and given a linear felicity function, and a normalization that y units of the consumption good yields a welfare level of y, $u(x,x')$ can be written as

$$\lambda(x,x')=\begin{cases}x & \text{if } x'\leq 1/a-\xi x\\ 1-a(x'-(1-d)x) & \text{if } x'\geq 1/a-\xi x\end{cases}$$

and we can now rewrite Section 2 with $\lambda(x,x')$ for $w(by)$. We also obtain that $(\hat{x},\hat{p})$ is given by $((1/2a),1/2)$. Furthermore, as proved in Khan and Mitra (2006, 2007), given the state today x, the state tomorrow minimizing the value-loss $\delta(x,x')$ is

$$x'=h(x)=\begin{cases}\frac{1}{a}-x & x\leq 1\\ (1-d)x & x>1\end{cases}$$

By the above it is easy to see that for any $x_0\in[1-d,1]$, the program

$$x^*(t)=x_0 \quad \text{if} \quad t=\dot{2} \quad \text{and} \quad x^*(t)=1/a-x_0 \quad \text{if} \quad t\neq\dot{2} \tag{2}$$

where $\dot{n}$ stands for a *multiple* of n, is the unique zero value-loss program originating from x_0. We see also that $y^*(t)=\lambda x^*$ is $y^*(t)=x_0$ if $t=\dot{2}$ and $y^*(t)=\frac{1}{a}-x_0$ if $t\neq\dot{2}$

Theorem 4.1. *There is no optimal program from $x_0\in[1-d,1]$.*

Proof. Due to Theorem 3.1, if there exists an optimal program it must be $\{x^*(t),y^*(t)\}$ defined in Eq. (2). We will see now that $\{x^*(t),y^*(t)\}$ cannot be optimal.

We study first the case where $x_0=\hat{x}$, the minimal value-loss program is of course, $x^*(t)=y^*(t)=\hat{x}$ for all t. Consider the following alternative program $x(0)=\hat{x}$, $x(1)=\hat{x}+\Delta x$ with $\Delta x\in(0,1-\hat{x}]$ and $x(t+1)=1/a-x(t)$ for all $t\geq 1$. It is trivial to check that $(x(t),x(t+1))\in\Omega$ for all t. This program yields a consumption $y(t)=\lambda(x(t),x(t+1))$,

$$y(t)=\begin{cases}1-a[\hat{x}+\Delta x-(1-d)\hat{x}]=\hat{x}-a\Delta x & t=0\\ x(t)=\hat{x}+\Delta x & t\neq\dot{2}\\ x(t)=\hat{x}-\Delta x & t=\dot{2},t>0\end{cases}$$

We easily get from there that

$$\sum_{t=0}^{T}\left[y(t)-y^*(t)\right]=\begin{cases}-a\Delta x<0 & T=\dot{2}\\ (1-a)\Delta x>0 & T\neq\dot{2}\end{cases}$$

and hence $\limsup_{T\to\infty}\sum_{t=0}^{T}[y(t)-y^*(t)]=(1-a)\Delta x>0$ and the program $\{x(t),y^*(t)\}$ is not optimal.

Assume now that $x_0\in(\hat{x},1]$, the minimal value-loss program is given by Eq. (2) with $y^*(t)=x^*(t)$, and consider the following alternative program $x(0)=x_0$, $x(t)=\hat{x}$ for all $t\geq 1$. This program yields a consumption $y(t)=\lambda(x(t),x(t+1))$ that is $y(0)=1-a[\hat{x}-(1-d)x_0]=1/2+(1-a)x_0$ and $y(t)=\hat{x}$ for all $t\geq 1$.

The difference of benefit at each stage is

$$y(t)-y^*(t)=\begin{cases}1/2+(1-a)x_0-x_0=1/2-ax_0 & t=0\\ \hat{x}-1/a+x_0=x_0-\hat{x} & t\neq\dot{2}\\ \hat{x}-x_0=\hat{x}-x_0 & t=\dot{2},t>1.\end{cases}$$

We easily get from there that $\limsup_{T\to\infty}\sum_{t=0}^{T}[y(t)-y^*(t)]=(1-a)(x_0-\hat{x})>0$ and hence the minimal value-loss program $\{x(t),y(t)\}$ cannot be optimal.

If $x_0 \in [1-d, \hat{x})$ consider the minimal value-loss program given by Eq. (2) and the following program $x(0)=x_0$, $x(1) = x^*(1) = 1/a - x_0 \in (\hat{x}, 1]$ and $x(t) = \hat{x}$ for all $t \geq 2$. The difference of benefit is

$$y(t)-y^*(t) = \begin{cases} 0 & t=0 \\ 1/2-ax_0 & t=1 \\ x_0-\hat{x} & t=\dot{2},\ t>1 \\ \hat{x}-x_0 & t\neq\dot{2},\ t>1 \end{cases}$$

Repeating the process above, we get $\limsup_{T\to\infty} \sum_{t=0}^{T}[y(t)-y^*(t)] = (1-a)(x_0-\hat{x}) > 0$. □

According to (Khan and Mitra, 2007a, Section 7), a complete description of the unique minimal value-loss programs originating from any given initial state is provided. Using this, we can present the following stronger result.

Theorem 4.2. *There is no optimal program from* $x \in \mathbb{R}_+$.

Proof. Given any $x \in \mathbb{R}_+$, let $\{x^*(t), y^*(t)\}$ be a minimal value-loss program from x. According to (Khan and Mitra, 2007a, Section 7), it is proved that the minimal value-loss program reaches the subinterval $[1-d, 1]$ in a finite number of steps and remains there following a 2-periodic cycle afterwards. Let T_x be the first stage such that $x^*(T_x) \in [1-d, 1]$.

Consider first the states such that $x^*(T_x) = \hat{x}$ and the following alternative program originating from it $x(t)=x^*(t)$ for all $t \leq T_x$, $x(T_x + 1) = \hat{x} + \Delta x$ where $\Delta x \in (0, 1-\hat{x}]$ and $x(t+1) = 1/a - x(t)$ for all $t > T_x$. By a procedure similar to the one of the theorem above, it is easy to see that the difference of benefit is

$$y(t)-y^*(t) = \begin{cases} 0 & t<T_x \\ -a\Delta x & t=T_x \\ \Delta x & t>T_x,\ t-T_x\neq\dot{2} \\ -\Delta x & t>T_x,\ t-T_x=\dot{2}. \end{cases}$$

Thus, $\limsup_{T\to\infty} \sum_{t=0}^{T}[y(t)-y^*(t)] = (1-a)\Delta x > 0$.

Consider now any $x \in \mathbb{R}_+$ such that $x^*(T_x) \neq \hat{x}$. We know from the previous results, that $\{x^*(t), y^*(t)\}$ will fulfill $x^*(t+1) = 1/a - x^*(t)$ for all $t \geq T_x$. Let an alternative program be

- if $\hat{x} < x^*(T_x) \leq 1 : x(t) = x^*(t)$ for all $t \leq T_x$ and $x(t) = \hat{x}$ for all $t > T_x$
- if $1-d \leq x^*(T_x) < \hat{x} : x(t) = x^*(t)$ for $t \leq T_x + 1$ and $x(t) = \hat{x}$ for all $t > T_x + 1$.

Repeating the procedure of the theorem above it is easy to see that in both cases

$$\limsup_{T\to\infty} \sum_{t=0}^{T}[y(t)-y^*(t)] = (1-a)(x_0-\hat{x}) > 0$$

and thus the unique minimal value-loss program cannot be optimal. □

5. Two open questions

As far as the non-existence of optimal programs in the RSS model are concerned, the example in the last section settles the question, and there is nothing more to be said.[10] However the line between a counterexample and a theorem is not always clear-cut, and the question that the results of this paper provoke is whether one can show, in the full generality of the GM model, that periodicity of a maximal program is necessary and sufficient for the non-existence of an optimal program from the same initial stock. Indeed, posing the question in this way, and taking the lead from Khan and Piazza (2009), one can ask whether the entire theory, including the conjectured non-existence result, can be investigated in the context of upper semicontinuous functions that can be supported at the *GRS*, rather than continuous concave functions. It is hoped that the overview of the work presented here, and written for a general audience, can help in the successful pursuit of these two questions, and thereby come up with a refashioned, and suitably generalized, GM model for the study of intertemporal allocation of resources.

[10] Subsequent to the writing of this note, Tapan Mitra shared with us another example showing that there is no optimal program from the golden-rule stock in the 2-sector RSS model.

Acknowledgements

This work was done when Khan held the position of Visiting Professor at the Nanyang Technological University, August, 2009. He thanks Professor Euston Quah and the Department of Economics at NTU for their hospitality, and Drs. Haifeng Fu and Zhixiang Zhang for extended conversation. Adriana Piazza gratefully acknowledges the financial support of FONDECYT under project 11090254 and that of Programa Basal PFB 03, CMM, U. de Chile.

References

Atsumi, H., 1965. Neoclassical growth and the efficient program of capital accumulation. Review of Economic Studies 32, 127–136.
Brock, W.A., 1970. On existence of weakly maximal programmes in a multi-sector economy. Review of Economic Studies 37, 275–280.
Cass, D., Stiglitz, J.E., 1970. The implications of alternative savings and expectation hypotheses for choices of technique and patterns of growth. Journal of Political Economy 77, 586–627.
Fujio, M., 2009. Optimal transition dynamics in the leontief two-sector growth model with durable capital: the case of capital-intensive consumption goods. Japanese Economic Review 60, 490–511.
Fujio, M., Khan, M. Ali, 2007. Ronald W. Jones and two-sector growth: Ramsey optimality in the RSS and Leontief cases. Asia-Pacific Journal of Accounting and Economics 13 (2006), 87–110.
Gale, D., 1967. On optimal development in a multi-sector economy. Review of Economic Studies 34, 1–18.
Khan, M. Ali, 2005. Intertemporal ethics, modern capital theory and the economics of forestry. Chapter 2. In: Kant, S., Berry, R.A. (Eds.), Economics, Sustainability, and Natural Resources: Economics of Sustainable Forest Management, Volumes 1 and 2. Springer, Berlin, pp. 39–65.
Khan, M. Ali, Mitra, T., 2005a. On choice of technique in the Robinson–Solow–Srinivasan model. International Journal of Economic Theory 1, 83–109.
Khan, M. Ali, Mitra, T., 2005b. On topological chaos in the Robinson–Solow–Srinivasan model. Economic Letters 88, 127–133.
Khan, M. Ali, Mitra, T., 2006. Undiscounted optimal growth under irreversible investment: a synthesis of the value-loss approach and dynamic programming. Economic Theory 29, 341–362.
Khan, M. Ali, Mitra, T., 2007a. Optimal growth in a two-sector RSS model without discounting: a geometric investigation. Japanese Economic Review 58, 191–225.
Khan, M. Ali, Mitra, T., 2007b. Optimal growth under discounting in the two-sector Robinson–Solow–Srinivasan model: a dynamic programming approach. Journal of Difference Equations and Applications 13, 151–168.
Khan, M. Ali, Mitra, T., 2008a. Growth in the Robinson–Solow–Srinivasan model: undiscounted optimal policy with a strictly concave welfare function. Journal of Mathematical Economics 44, 707–732.
Khan, M. Ali and Mitra, T., 2008. Bifurcation Analysis in the Two-Sector Robinson–Solow–Srinivasan Model: Notes on the Outside Case. Mimeo, Cornell University.
Khan, M. Ali, Piazza, A., 2009. On the Mitra-Wan Forestry Model: A Unified Analysis. *Publicación Técnica CMM* No. 226. Paper presented at *Workshop in Economic Dynamics*, National University of Singapore, July 30 to August 1, 2009.
Khan, M. Ali, Zaslavski, A.J., 2007. On a uniform turnpike of the third kind in the Robinson–Solow–Srinivasan model. Journal of Economics 92, 137–166.
Khan, M. Ali, Zaslavski, A.J., 2008. On existence of optimal programs: the RSS model without concavity assumptions on felicities. Journal of Mathematical Economics 46, 1–14.
Khan, M. Ali, Zaslavski, A.J., 2010a. On two classical turnpike results for the Robinson–Solow–Srinivasan (RSS) model. Advances in Mathematical Economics 13, 47–97.
Khan, M. Ali, Zaslavski, A.J., 2010b. On locally optimal programs in the RSS (Robinson–Solow–Srinivasan) model. Journal of Economics 99, 65–92.
McKenzie, L.W., 1986. Optimal economic growth, turnpike theorems and comparative dynamics. In: Arrow, K.J., Intrilligator, M. (Eds.), Handbook of Mathematical Economics, Vol. III. North-Holland, Amsterdam, pp. 1281–1355.
McKenzie, L.W., 2002. Classical General Equilibrium Theory. The MIT Press, Cambridge, MA.
Metcalf, C., 2008. The dynamics of the Stiglitz Policy in the RSS model. Chaos, Solitons & Fractals 37, 652–661.
Mirrlees, J.A., 1962. Choice of techniques. Indian Economic Review 6, 93–102.
Nishimura, K., Yano, M., 1995. Nonlinear dynamics and chaos in optimal growth: An example. Econometrica 63, 981–1001.

Okishio, N., 1966. Technical choice under full employment in a socialist economy. The Economic Journal 76, 585–592.

Peleg, B., 1973. A weakly maximal goldn-rule program for a multi-sector economy. International Economic Review 14, 574–579.

Robinson, J., 1960. Exercises in Economic Analysis. MacMillan, London.

Solow, R.M., 1962. Substitution and fixed proportions in the theory of capital. Review of Economic Studies 29, 207–218.

Srinivasan, T.N., 1962. Investment criteria and choice of techniques of production. Yale Economic Essays 1, 58–115.

Stiglitz, J.E., 1968. A note on technical choice under full employment in a socialist economy. The Economic Journal 78, 603–609.

Stiglitz, J.E., 1973. Recurrence of techniques in a dynamic economy. In: Mirrlees, J., Stern, N.H. (Eds.), Models of Economic Growth. John-Wiley and Sons, New York.

von Weizsäcker, C.C., 1965. Existence of optimal programs of accumulation for an infinite time horizon. Review of Economic Studies 32, 85–104.

Zaslavski, A.J., 2005. Optimal programs in the RSS model. International Journal of Economic Theory 1, 151–165.

Zaslavski, A.J., 2007. Turnpike results for discrete-time optimal control systems arising in economic dynamics. Nonlinear Analysis 67, 2024–2049.

Zaslavski, A.J., 2010. Good locally maximal programs for the Robinson-Solow-Srinivasan model. Advances in Mathematical Economics 13 (Part 1), 161–176.