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Optimal cyclicity and chaos in the 2-sector RSS model: An anything-goes construction☆

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ABSTRACT

We present a simple and transparent construction that furnishes, for any pre-chosen dynamic, particular instances of the 2-sector version of the Robinson–Solow–Srinivasan model that yield the chosen dynamic, including optimal topologically chaotic programs and those that exhibit cycles of *any* given period. The construction is expressed in terms of ξ, the marginal rate of transformation of capital from one period to the next with zero consumption, an important summary statistic of the model discovered by Khan and Mitra. Our construction relies on theorems due to Li–Yorke and Sharkovsky, and complements earlier work on chaotic dynamics in the RSS model.

☆ This essay comes out of an ongoing research project initiated during Khan's visit to the Department of Economics and the *Center for Analytical Economics (CAE)* at Cornell University in July 7–14, 2004, and continued during Khan's visit to the *Centro Modelamiento Matematico (CMM)* at the Universidad de Chile, Santiago, December 26, 2005 to January 11, 2006. A preliminary report, with Professor Tapan Mitra as co-author, was written up during Khan's subsequent visit to Ithaca in June 2006, and circulated under the title "The Li–Yorke theorem and topological chaos in the 2-sector RSS model." The authors thank him, Alejandro Jofré, Ashvin Rajan, Barkley , Santanu Roy and David Wiczer for stimulating conversation and encouragement. They also thank the *CMM* and *CAE* for research support. Adriana Piazza gratefully acknowledges the financial support of Programa Basal PFB 03, *CMM*, and that of FONDECYT under Project 11090254.

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Mathematics, rightly viewed, possesses not only truth, but supreme beauty, capable of a stern perfection such as only the greatest art can show.
(Bertrand Russell, 1910)[1]

The classical goal of geometry is the exploration and enumeration of geometric configurations of all kinds. . . . to discover perfect objects. This is a flower garden whose beauty has almost been forgotten in the 20th century rush to abstraction and generality.
(David Mumford, 2009)[2]

Mathematical objects are a special variety of a social-cultural-historical object. . . a shared idea, like Moby Dick in literature, or the Immaculate Conception in religion.
(Borwein and Bailey, 2008)[3]

The 'real world' may or may not be simple. If it is, so much the better. If, as is more likely, it is not, then we still need, and have, sophisticated analytical tools to study it. But the values of the parameters determining the comparative statics predictions have to be calibrated by an empirical appeal.
(Andreu Mas-Colell, 1989)[4]

1. Introduction

It is now clear that the general theory of intertemporal allocation of resources was transformed and given an additional, if not totally novel, direction in the mid-eighties: rather than the global turnpike, interest shifted to the neighborhood turnpike, and it was shown that optimal cyclical and chaotic phenomena were the rule for "small" discount factors in a variety of deterministic infinitely lived representative agent economic models. The theory is comprehensively surveyed[5] in Dana et al. (2006), and the important earlier papers have been collected in two anthologies, Benhabib (1992) and Majumdar et al. (2000), with the latter also providing suggested outlines for possible courses in "dynamic optimization" and "chaotic models in economic theory."[6] The basic outline of the theory, and in particular, its precise formalization of "small" discount factors is then well-documented and well-understood.[7]

The essential highlights of this narrative can be simply and clearly identified. Boldrin and Montrucchio (1986a,b) and Denekere and Pelikan (1986) showed that any twice continuously differentiable function can be rationalized as the policy function of an appropriately defined dynamic-optimization model.[8] Since the logistic map $h : x \to 4x(1-x)$, $x \in [0, 1]$, is twice continuously differentiable, and it yields complicated dynamics, these results made it clear that such trajectories would have to be accommodated by the theory. However, in some sense these results were stronger than was really needed. Given the singular role that the work of Sharkovsky (1964) and Li and Yorke (1975) assigned to 3-period cycles for one-dimensional systems,[9] all that had to be shown was that a function yielding such behavior, the tent-map for example, could be rationalized as the policy function of an appropriately defined and mainstream dynamic optimization model. This was established by Boldrin (1989), Boldrin and Deneckere (1990) and Nishimura and Yano (1994, 1995). The issue then devolved as to the numerical meaning that could be given to the relevant notion of a "small" discount factor. Following on important initial work by Sorger (1994a,b) and Montrucchio (1994), a "definitive" answer was given by Mitra (1996) and Nishimura and Yano (1996), definitive in the sense that the "universal constant" $[(\sqrt{5}-1)/2]^2$ was seen to be necessary and sufficient for a 3-period cycle. Further refinements pertaining to turbulence were furnished by Mitra (2000b).[10]

[1] From *The Study of Mathematics: Philosophical Essays,* p. 73. Russell elaborates: "a beauty cold and austere, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure,. . ."

[2] Mumford is commenting on the work of the Canadian geometer H. M. S. Coxeter; private communication quoted in Borwein and Bailey (2008, p. 87). Like Russell, Mumford also elaborates, though with a slippage to a psychological register: "The goal is not especially to prove theorems but to discover perfect objects: theorems are only a tool that imperfect humans need to reassure themselves that they have seen these objects correctly." For another invocation of the psychological register, see Mas-Colell (1989; p. 507).

[3] This is a paraphrase of Reuben Hersh in "Fresh breezes in the philosophy of mathematics"; see Borwein and Bailey (2008, p. 41).

[4] This turn from mathematics to mathematical economics, more a grounding than a slippage, is expressed in an essay titled "Capital theory paradoxes: anything goes"; see Mas-Colell (1989; pp. 506–507). This fourth epigraph to this essay is also intended as a (token) appreciation of the inspiration they have received over the years from Mas-Colell's evolving *oeuvre.*

[5] See Chapters 4 and 6 in Dana et al. (2006).

[6] In addition to Chapters 3 and 12 in Majumdar et al. (2000), and Chapters 1, 4 and 6 in Dana et al. (2006), see McKenzie (1986, 2002) for references and details on the neighborhood turnpike theorem, a basic result of the subject. Also see Majumdar (2009; Chapter 3) for an affiliated but different direction.

[7] One may perhaps mention here (the so-called) McKenzie's folk-theorem, as in McKenzie (1983), with further elaborations in Khan (2006); Khan and Mitra (2007c). Active research continues of course, as can be seen from the papers in Mitra and Nishimura (2001) and the references in Khan and Mitra (2006b, 2007b), Liu (2009) and Footnote (19) below; we would not be writing this essay if that were not so.

[8] Also see antecedent work by Montrucchio (1984), referred to in Boldrin and Montrucchio (1986a), and in Montrucchio (1988).

[9] See the expositions of these results in Benhabib (1992) and Majumdar et al. (2000). In the mathematical literature, see for example, Block (1986), Block and Coppel (1991), Misiurewicz (1995), Sharkovsky et al. (1997), Nagashima and Baba (1999) and Brucks and Bruin (2004). Misiurewicz (1995), in particular, has a brief and accessible discussion regarding the extension to two-dimensional systems.

[10] For the precise meaning given to necessity and to sufficiency, see the two papers referenced above, and the recent survey by Sorger, presented as Chapter 4 in Dana et al. (2006).

From the viewpoint of this essay, two different methodological directions in this work can be distinguished: one direction constructs economies of a particular class and type that exhibit a pre-chosen dynamic; and the other, characterizes completely, or at least as completely as possible, the resulting dynamics of a particular well-specified, explicitly parameterized model. Since the one-sector (Ramsey) optimal growth model yields complicated dynamics only when the felicity function exhibits wealth effects, as emphasized and studied by Majumdar and Mitra (1994), it is not surprising that under both directions, the 2-sector model has played a rather prominent, if not decisive, role in this reorientation of the general theory. Thus Nishimura and Yano (1994, 1995, 2000), in work that is an exemplar of the first set of papers, work with a 2-sector Leontief setting;[11] while Boldrin and Deneckere (1990), in work that exemplifies the second set,[12] investigate a 2-sector model with a Cobb–Douglas and Leontief production functions. Work under either rubric has been conducted with the understanding, emphasized early on by May (1976), and subsequently by Saari (1995), that the simpler the underlying model, the stronger the result: one strives for the simplest model that can rationalize a pre-chosen dynamic, or the simplest prototypical model that yields the entire range of cyclic and chaotic phenomena.[13]

In this decade of work, essentially from the mid-eighties to the mid-nineties, the relevance of a special case of the 2-sector model seems to have been missed. This is the so-called 2-sector RSS model that emerged as a special case of a model due to Robinson, Solow and Srinivasan, one originally formulated to investigate questions relating to the “choice of technique” in development economics.[14] This is a model in which one of the 2-sectors is Leontief, as in the earlier work referred to above, but the other is simply a linear (Ricardian) function of only one factor, labor. It is a model that activates the labor theory of value.[15] Thus, rather than a full 2-sector model, it is a special case of the Leontief–Shinkai model in which one of the two outputs is produced by only one of the two factors.[16] If the Leontieff coefficients, the coefficient of the linear felicity function, and the stationary labor stock are all normalized to unity, the model can be reduced to a parametric specification given by the triple (a, d, ρ), $a > 0$, the amount of labor required to produce a unit of a perfectly divisible machine, d the depreciation rate, and ρ the discount factor, the last two positive reals lying in the unit interval. Whereas a full characterization of the optimal dynamics of such a model has yet not been achieved in the discounted case, it has been shown to exhibit phenomena not present in an identical model but in continuous time: optimal 2- and 4-period cycles, coincident optimal policy correspondences in the undiscounted and discounted cases for a non-negligible range of the discount factor, the existence of optimally chaotic trajectories for “small” discount factors. In particular a parameter ξ, $-1 < \xi < \infty$, has been identified: it formalizes the marginal rate of transformation from the amount of machines this period to the next but with zero consumption, and serves as a sufficient marker for an identification of the full variety of optimal economic dynamics in the 2-sector RSS model in both the undiscounted and discounted settings.[17]

All of this work has been conducted under what we are now identifying here as the second rubric: a research program devoted to a complete charting out of the long-run and transition dynamics in both the undiscounted and discounted settings of the 2-sector RSS model, a model that, given its simplicity and analytical (indeed geometric) tractability, has the potential to serve as a prototypical illustration of the salient points of the general theory. In this essay, we continue, in the first place, this direction of work by complementing recent results on the existence of optimal topological chaos in the 2-sector RSS model; see the survey in Dana et al. (2006; Chapter 6). Since a brief recapitulation of these results is necessary to place the contribution in perspective, we turn to it.

Khan and Mitra (2005b) discover optimal topological chaos in the 2-sector RSS model by appealing to results in the theory of turbulence, as exposited in Block and Coppel (1992). They offer a particular example of the model, one in which a and d (alternatively, a and ξ) satisfy the restriction

$$(1-d)\left(\frac{1}{a}-\xi(1-d)\right)=\hat{x} \Leftrightarrow \left(\frac{1}{\xi}-\xi\right)(1-d)=1 \Leftrightarrow a=\frac{\xi^2-1}{\xi^3}, \tag{1}$$

[11] It is to be noted, however, that the work of Nishimura and Yano, with its powerful geometric conception, has been carried forward in Fujio (2005, 2008, 2009), under the rubric of what we are characterizing the second direction.

[12] See paragraph 4 in Boldrin and Deneckere (1990), for example. Majumdar and Mitra (1994) write in their introduction, “The approach... is somewhat different from... the literature. We start with a model of dynamic optimization, specified in terms of the primitives [and ask] what kinds of dynamic optimal behavior can this model exhibit?”

[13] Thus Majumdar and Mitra, Chapter 7 in Majumdar et al. (2000), write, “Besides the *relative abundance* of examples of chaos, yet another theme has been rightly stressed: *quite simple* models of economic theory may lead to such examples.” For a quotation from Saari, see Rosser (1999), a reference that also places Mandelbrot (1983) in the general corpus of work).

[14] See Stiglitz (1968) and Khan and Mitra (2003); ? for genealogical details and initial references, especially to those of Joan Robinson. Also see Metcalf (2008) for a comprehensive numerical analysis. Two references that seem to have been missed, however, are Little (1957) and Robinson (1974). Furthermore, as we shall see below in Section 3.2, Boldrin and Montrucchio (1986a) consider a dual model, one in which machines constitute the only factor used in both sectors. It should be noted that the model, despite its two sectors, has only one capital good and therefore its dynamics fully delineated by a single difference equation.

[15] This was realized early on; see Stiglitz (1968), and the references in Khan and Mitra (2005a) to the Robinson–Stigltiz exchange and to subsequent work arising from it. Also see an early analysis by Rosser (1983).

[16] For this precursor of the Uzawa–Srinivasan model, and for a justification of the terminology, see Fujio (2005). The model in Rosser (1983) is a further generalization.

[17] See, in particular, Khan and Mitra (2007a, 2006a,b) and their references. This parameter makes no appearance in the earlier analyses; and especially in that of Stiglitz (1968) conducted in continuous time. Also see Khan and Mitra (2003).

where $\hat{x} \equiv 1/(1+ad)$ is the modified golden-rule stock,[18] and $\xi \equiv (1/a) - (1-d)$ is the marginal rate of transformation referred to above. The fact that such an example exists, which is to say that there are a and d (alternatively, a and ξ) satisfying (1), is established through an application of the intermediate-value theorem. Once it is shown that a program converging to the modified golden-rule stock in three periods is indeed optimal, an appeal to a theorem of Misiurewicz completes the demonstration of the existence of optimal chaotic programs. This invokes the sufficient condition $\rho < a$, $0 < a < 1$, a condition that forced a policy correspondence to be a function (its continuity is a routine consequence of Berge's maximum theorem), and formalized the requirement that the discount factor be "small". Apart from its substantive interest, the result is also of methodological interest in that in addition to continuity, its proof does not appeal to any unimodel or other properties of the policy function. As such, optimal (topological) chaos can be established even when the question of the shape of the optimal policy function remains unsettled.

One of the motivating questions of this essay then is whether optimal topological chaos can also be established by an appeal to the simpler Li–Yorke theorem (1975) (also see May (1976) for further exposition) under the same methodological constraint that the exact shape of the policy function is unknown? More specifically, does there exist a parametric example with a continuous policy function for which a 3-period cycle constitutes an optimal program? We show that this is indeed the case. It is one in which a and d (alternatively, a and ξ) satisfy the restriction

$$(1-d)\left(\frac{1}{a} - \xi(1-d)\right) = 1 \Leftrightarrow (\xi - 1)(1-d) = 1 \Leftrightarrow a = \frac{\xi - 1}{\xi^2 - \xi + 1}. \tag{2}$$

Such an example arises as a result of geometric considerations related to perpendicularity,[19] and leads to a unified geometric depiction of the two cases delineated in (1) and (2) above. What is interesting, and seems to have been missed in earlier work, is that with this geometric representation of the first example, additional features come to light whereby not only further computations are rendered unnecessary, but that the bound of the discount factor can be raised in the second example from a to $(1/\xi)$, and shown to be a least upper bound. Thus, in addition to the relevance of the Li–Yorke theorem, we can also present a simpler proof of a considerably strengthened result.[20]

Whereas a search for the second example constituted an original motivation of this essay, and the simplification of the first its unintended byproduct, we are clear that these results do not represent its primary contribution. The existence of chaotic trajectories is, in and of itself, not especially surprising two decades later than they were originally established, and even in the case of the 2-sector RSS model, something already shown to be possible in Khan and Mitra (2005b).[21] The primary contribution of this essay is perhaps to the body of work discussed in this introduction under the first rubric – the construction of a simple economy to exhibit a pre-specified optimal dynamic. We can focus on the essentials of the argumentation referred to above to show that for any given value of ξ greater than unity, one can simply and transparently construct, essentially with compass and straight-edge,[22] a 2-sector RSS model (economy), which is to say a triple (a_ξ, d_ξ, ρ_ξ), that yields the type of transition dynamics that is desired: starting from an initial capital stock of unity, a 2-, 3- or 4-period cycle or convergence to the golden rule stock in 3 periods, or indeed, for optimal trajectories of *any* periodicity! The constructive procedure that we identify, in relying only on the magnitude of ξ, the sufficient statistic mentioned above, stands in useful comparison with analogous constructions in Boldrin and Montrucchio (1986a,b) and Nishimura and Yano (1994, 1995, 2000).[23] In particular, it leads us to ask for the smallest value of ξ that yields a pre-chosen optimal dynamic of the 2-sector RSS model. We show that, starting from an initial unit stock of capital, such a value of ξ is unity for the existence of a 2-period cycle, greater than unity for a 4-period cycle, two for a 3-period cycle, and above all, the golden number $[(1+\sqrt{5})/2]$ for convergence to the golden-rule stock in 3-periods.[24] Since preliminary analysis suggests $(1/\xi)$ as a particularly important discount factor, this leads to the value $[(\sqrt{5}-1)/2]$. Thus, this construction is of independent potential interest, and inspite of the discussion presented in the sequel, its full implications are far from exhausted.

Thus, in keeping with the title of this essay, we present an "anything goes" construction that brings to mind an affiliated inquiry into capital theory initiated by Mas-Colell (1989), one also involving the work of Joan Robinson.[25] However, rather than an economy with a pre-chosen optimal dynamic as we do here, Mas-Colell considered an economy with a pre-chosen

[18] Precise definition of this standard concept to be furnished below.

[19] A point of view explicitly exploited in Khan and Mitra (2006a,b, 2007b). For a detailed further (non-geometric) analysis of the equalities in (1) and (2), see ongoing work being pursued by Khan and Mitra in a preliminary 2009 manuscript titled "Complicated dynamics and parametric restrictions in the Robinson–Solow–Srinivasan Model."

[20] Strengthened in the sense that, unlike the first example, it exhibits a 3-period cycle.

[21] The authors thank Tapan Mitra for his emphasis on this fact.

[22] As is well-known to the *cognoscenti,* this is a time-honored imperative for geometric labor: see Kostovskii (1961); Coxeter and Greitzer (1967); Kazarinoff (1970) and their references.

[23] We return to this work, along with that of Boldrin and Montrucchio (1986a,b) in the sequel.

[24] For discussion of this number, also called *phi*, see Huntley (1970) and Livio (2003); and in the context of mathematical economics, Footnote 3 in Mitra (1996).

[25] Mas-Colell wrote "My intention is to make Joan Robinson's point in the manner of the Sonnenschein–Mantel–Debreu theorem, to out-Cambridge Cambridge, so to speak."

comparative-static result, albeit in a steady-state dynamic context.[26] In a general capital-theoretic model, "given any set of consumption and rate of interest pairs," he affirmed that "a well-behaved technology can be found having precisely this set as the steady-state comparative static locus." From a modeling rather than a methodological point of view, it is perhaps worth emphasizing that in contrast to Mas-Colell's inquiry, ours is conducted in a model with a single capital good.

> In capital theory, only the one capital good case is amenable to self-evident graphical analysis. The interesting complexities arise, however, only with more then one capital good.[27]

The analysis reported in this essay, as well as that conducted in the earlier references, while certainly underscoring Mas-Colell's first sentence, deflects his second sentence to a number of sectors from a number of capital goods.[28]

One final introductory observation in this connection relates to the unexpected similarity of the construction reported here to that of Denekere and Judd (1992) in their investigation of a model of innovation with a continuum of a possible number of commodities. After a scale normalization, our construction proceeds in a square of size ξ, and bears an uncanny resemblance to what they call their "trapping region," again a square of normalized size, and which they transform, in keeping with their analysis of a given model and their focus on ergodic chaos, by a 180° rotation. A detailed investigation of their technical apparatus and its comparison with that of ours, and the identification of insights beneficial to both, is beyond the scope of this essay,[29] but it does lead us to make two observations of a methodological nature: first, the distinction between the two different branches of the literature that we categorize should not be overemphasized; and second, the importance of geometry as a complementary engine of analysis perhaps ought not to be overlooked as much as it has so far been.[30]

We conclude this introduction by underscoring the fact that the above narrative, concrete and complete as it is, is but a Ramseyian instance in the larger narrative of economic dynamics that has unfolded since the early eighties. This literature is not always hinged to intertemporal optimization of a single planner or representative agent, and is thereby more directly related to the mathematical literature in that it does not insist that the solutions to differential and difference equations be in addition, maximizers of an intertemporal utility function.[31] The distinctive thrust of this literature, as opposed to the offshoot that is the concern of this essay, goes back to the (now old-fashioned)[32] distinction between "descriptive and optimal growth theory" and its implications for the work on rational expectations models is examined in Rosser's (1999) survey. We shall return to this broader literature in the third of our concluding remarks.

2. The 2-sector RSS model and its antecedents

The basic contours of the 2-sector RSS model have already been presented in the introduction. There is a consumption and an investment sector, and at each time stage, the planner allocates among them an amount of perfectly divisible capital and an exogenously given, also perfectly divisible, amount of labor. This, after taking an exogenously given rate of capital depreciation into account, yields a new level of capital in the next period, which allows a repetition of the process. Given that the planner's objective is to maximize the discounted sum of aggregate consumption, Ramsey's (1928) question can be posed and studied.

Let the total labor force of the economy be stationary and positive, we normalize it to be unity. Machines are produced using only labor: $a > 0$ units of labor are needed to produce 1 unit of machine. If $x(t)$ is the amount of machines available this period, $x(t+1)$ the amount of machines available in the next period, and $d \in (0, 1)$ the rate of capital depreciation, we obtain

$$x(t+1) \geq (1-d)x(t)\,(\text{irreversibility constraint}),\ a(x(t+1)-(1-d)x(t)) \leq 1(\text{the labor constraint}). \tag{3}$$

These two constraints represent the stationary technology Ω in the space of today and tomorrow capital stocks (x, x'), and are respectively pictured in Fig. 1 as the lines OD and VL.

[26] Mas-Colell sights, in a move reminiscent of Samuelson's "correspondence principle", comparative statics as being closely related to the topic of stability; see Mas-Colell (1989; p. 506). Of course, the original motivation for the Sonnenschein construction was also stability of competitive equilibria; on this, see Majumdar (2009; Chapter 3). For another recent direction in this general enquiry, see Opp et al. (2009) and their references to the earlier literature.

[27] See Mas-Colell (1989; p. 507); the point is being made with respect to the comparison with Edgeworth box and general equilibrium theory.

[28] We take this opportunity to point out that Mas-Colell's (1989) theorem, in its reliance on differentiability of its reduced-form utility function ($v(\cdot,\cdot)$ does not apply to the 2-sector RSS model being considered in this essay.

[29] However, see Khan and Linask (2009) and their references to the pioneering work of Day and Shafer.

[30] In connection with the first observation, see the work of Fujio referenced in Footnote (11), and in connection with the second, in addition to our second epigraph, see Footnote (22).

[31] See, for example Day and Chen (1993); Day (1994); Goodwin (1990); Kelsey (1988), Rosser (1991, 1999, 2000) and their references. Rosser's (2000) text, in particular, contains a bibliography of around 1200 items spread over 75 pages. To be sure, some of the important papers are reprinted in Benhabib (1992). For the relevant mathematical texts, also see Footnote (9) above.

[32] This distinction is muted in Solow's (1999) survey where optimal growth theory is subsumed within the general rubric of neoclassical growth theory. It is the text of Majumdar et al. (2000), and now of Dana et al. (2006), that is explicitly focused on optimization. Work such as that of Day and Huang (1990) and Kaas (1998) can be located as being in an intermediate space. For a recent paper along this line and in this space, see Dudek (2010).

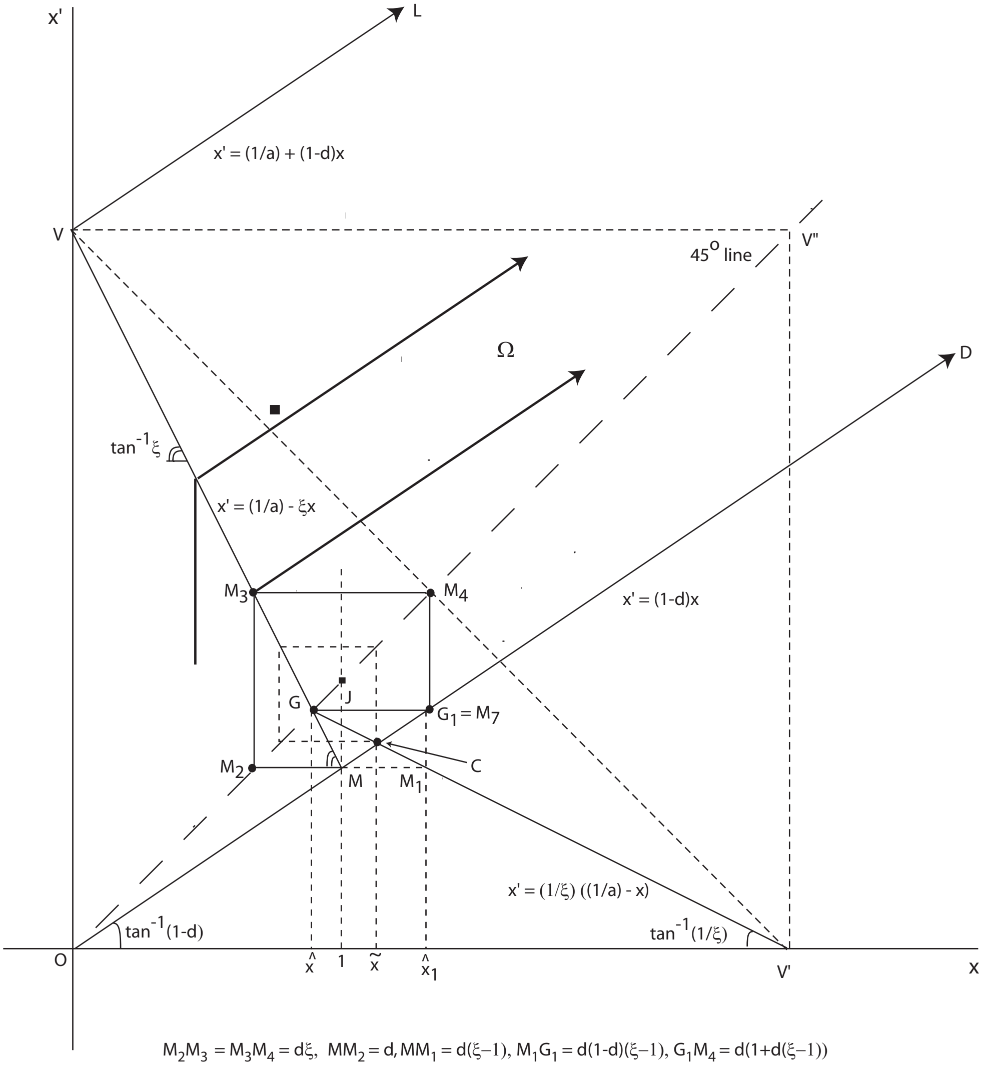

Fig. 1. Basic geometrical benchmarks of the 2-sector RSS model: The case $a\xi^3 = (\xi^2 - 1)$ or $(\xi - (1/\xi))(1 - d) = 1$.

One unit of labor together with one unit of machine, can produce 1 units of consumption good, thus, given the pair $(x(t), x(t+1)) \in \Omega$, the set of consumption levels available each period must fulfill

$$0 \le y(t) \le x(t)\,\text{(the capital constraint)},\;\; y(t) \le 1 - a(x(t+1) - (1-d)x(t))\,\text{(the labor constraint)}. \quad (4)$$

Thus, consumption $y(t)$ that is obtained is given by max $[x(t), 1 - a(x(t+1) - (1-d)x(t))]$. In Fig. 1, the indifference curves of the parameterized reduced-form felicities are given by the kinked lines “pegged” by the *MV* line with (absolute value of the) slope ξ being $(1/a) - (1-d)$. The zero and unit indifference curves are given by *OVL* and *DM*1, respectively.

It is by now well-understood that it is the *MV* line that makes geometry a viable analytical device for the analysis of the 2-sector RSS model.[33] To see this, we turn to a crucial benchmark in the general theory on intertemporal optimal allocation: the notion of a *modified golden-rule stock*. If $u : \Omega \to$ I\kern -1.7pt R is the reduced form felicity function,[34] such a stock $\hat{x}$ is formally defined as

$$u(\hat{x}, \hat{x}) \ge u(x, x')\,\text{for all}\,(x, x') \in \Omega\,\text{such that}\,x \le (1-\rho)\hat{x} + \rho x'. \quad (5)$$

A distinctive feature of the RSS model with discounting is that a (unique) modified golden-rule stock is independent of the discount factor even when it takes the unit (undiscounted) value. Given the specification of the indifference curves of the

[33] This is perhaps the primary methodological contribution of Khan and Mitra (2006a, 2007a).

[34] The function $u(x, x')$ is assumed to be upper semicontinuous, concave on Ω and increasing in its first argument and decreasing in its second argument.

reduced form utility function, it is easy to check that the modified golden-rule stock is given by the point G in Fig. 1 and that a fixed point problem has been converted to an optimization problem. Since Brock (1970), the importance of the golden-rule stock and its price-support by the golden-rule prices is well-understood for the undiscounted setting of the theory.[35] In terms of the parameters of the 2-sector RSS model, we obtain the following:

Proposition 1. *A unique modified golden-rule stock of the economy E is given by* $\hat{x} = 1/(1 + ad)$ *and it is price-supported by* $\hat{p} = 1/(1 + \rho\xi)$. *This is to say that* $(\hat{x}, \hat{x}) \in \Omega$, *and*

$$u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} \geq u(x, x') + \hat{p}(\rho x' - x) \text{ for all } (x, x') \in \Omega.$$

This leads to the definition of the value-loss of a production plan $(x(t), x(t+1)) \in \Omega$ relative to $(\hat{x}, \hat{p})$, by

$$\delta^{\rho}_{(\hat{x},\hat{p})}(x(t), x(t+1)) \equiv \delta(t) = (\frac{\rho\hat{p}}{a}) - u(x(t), x(t+1)) - \hat{p}(\rho x(t+1) - x(t)),$$

and allows the optimality criterion to be one based on minimizing aggregate value-losses. This is to say

$$\sum_{t=0}^{\infty}\rho^t[u(x'(t), x'(t+1)) - u(x''(t), x''(t+1))] = \sum_{t=0}^{\infty}\rho^t[y'(t) - y''(t)] = \sum_{t=0}^{\infty}\rho^t[\delta'(t) - \delta''(t)], \quad (6)$$

where $\{x'(t), y'(t)\}$ and $\{x''(t), y''(t)\}$ are two programs starting from the same initial stock. Furthermore, it is not only that the MV line is the zero value-loss line, but that parallel lines to it are iso-value-loss lines, increasing in value-loss as they move outwards in both the northeastern and southwestern directions from the MV line. The actual value loss is measured relative to a line of slope $(1/\rho)$ passing through the point G rather than the 45° line as in the undiscounted case.[36]

We can now conclude this section by specifying an *economy E* to consist of a triple (a, d, ρ), $0 < \rho < 1$, the discount factor, and to listing the following concepts that apply to it. A *program from* x_o is a sequence $\{x(t), y(t)\}$ such that $x(0) = x_o$, and $(x(t), x(t+1)) \in \Omega$, and $y(t)$ satisfies (4) for all $t = 0, 1, \cdots$. A *program* $\{x(t), y(t)\}$ is simply a program from $x(0)$, and it is said to be *stationary* if $(x(t), y(t)) = (x(t+1), y(t+1))$ for all $t = 0, 1, \cdots$. A program $\{x^*(t), y^*(t)\}$ from x_o is called *optimal* if $\sum_{t=0}^{\infty}\rho^t[c(t+1) - c^*(t+1)] \leq 0$ for every program $\{x(t), y(t)\}$ from x_o. A *stationary optimal program* is a program that is stationary and optimal.

3. The construction: a first pass

It is clear from Figs. 1 and 3, and earlier work, that the geometric ingredients of the 2-sector RSS model are given by the 45° line, the OD line representing the depreciation rate d, the unit capital stock designated by the point M (with coordinates $(1, 1-d)$), the MV line representing the composite parameter ξ, with its intersection V with the x' axis, delineating $(1/a)$. The analysis proceeds by completing the square $OVV''V'$ of size $(1/a)$, and by joining the point V' to the point G to obtain the GV' line, one that can be seen as the dual to the MV line by virtue of the slopes of the two lines summing to a right angle. The intersection of GV' with OD yields the important point C. We now turn to use these benchmarks to provide a simple and useful construction for optimal dynamics in the 2-sector RSS model.

3.1. The basic geometry

The construction that we report in this subsection isolates the smaller square $M_1M_2M_3M_4$ of size $d\xi$ within $OVV''V'$. We focus on it to substantiate the following claim.

Proposition 2. *Corresponding to any* $\xi > 2$, *there exists* $d_\xi = (\xi - 2)/(\xi - 1)$ *and* $a_\xi = (1/(\xi + (1 - d_\xi))$, *and any* $\rho_\xi < a_\xi$, *such that the 2-sector RSS economy* $E_\xi = (a_\xi, d_\xi, \rho_\xi)$ *yields a pre-chosen dynamic in terms of optimal 2- or 3- or 4 cycles and one that leads to optimal convergence to the golden-rule stock in 3-periods when starting from a unit capital stock.*

We now present a constructive proof of this claim. In Fig. 2, consider a vertical segment MM_{11} of (normalized) unit length. Given $\xi > 1$, draw a horizontal segment $M_{11}M_5$ of length $(1/\xi)$ at M_{11}. Join M to M_5, and extend this line outwards; it'll constitute the MV line of the specific instance of the model we are in the process of constructing.[37] Note that so far our procedure can be alternatively phrased as the simple delineation of the angle $\tan^{-1}(1/\xi)$ to MM_{11} at M.

Bisect the 90° angle at M_{11} to obtain the 45° line, with G as its point of intersection with MM_5. Let the intersection of this 45° line with the horizontal through M be M_2, and let the upward vertical through M_2 intersect the extension of the segment MM_5 at M_3.

Next, with the help of the 45° line, complete the square $M_{11}M_5M_9M_{10}$. Let GM_{10} meet the horizontal through M at M_1; GM_1 is the dual MV line. Let the upward vertical at M_1 intersect the 45° line at M_4, the horizontal M_5M_{11} at M_6, the

[35] See the first three paragraphs of Mas-Colell's (1989) introduction.

[36] For a substantiation of these claims, see Khan and Mitra (2006a,b) which extends Khan and Mitra (2007a) to the discounted case.

[37] The notation for the designation of the individual points may seem arbitrary to the reader but it is an attempt to conform to that already established in Fig. 1.

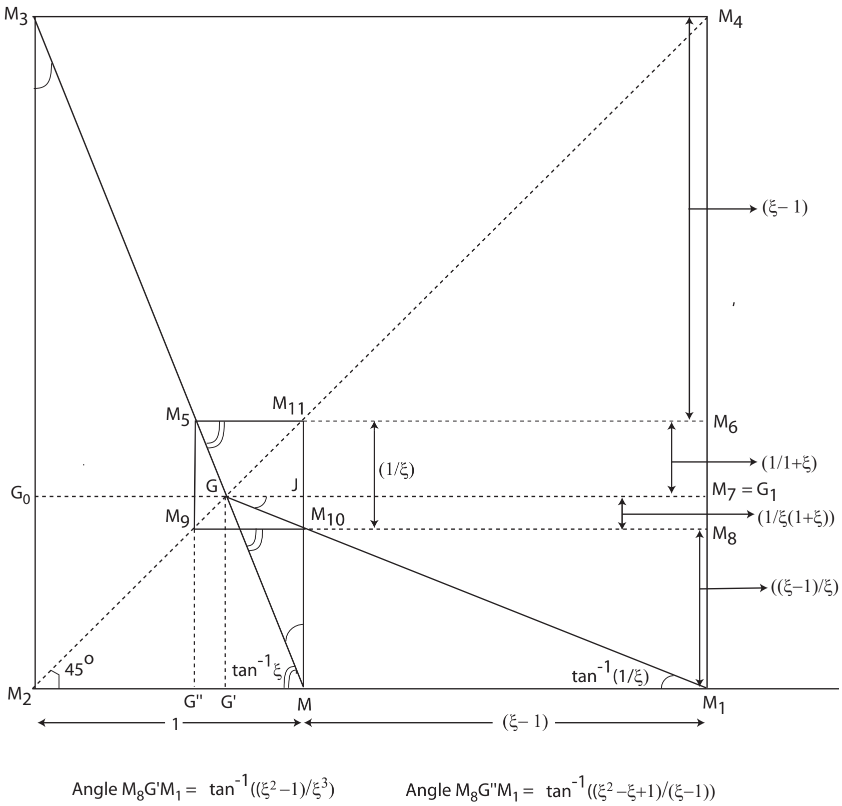

Fig. 2. The basic construction for $\xi > 1$.

horizontal through G at M_7 and the horizontal M_9M_{10} at M_8 – these last three points will serve as important benchmarks. Note that the square $M_1M_2M_3M_4$ is of size ξ – the fact that it is a square is evident from the delineation of the 45° line. Let the points G_0 and J be as specified there in Fig. 2. The construction is complete.

The key to the entire construction is the fact that the slope of the GM_1 line is $(1/\xi)$. To see this, simply observe that the triangles GG_0M_3 and $GG'M_1$ are congruent: the segments GG' and GG_0 are equal, and given that $M_1M_2M_3M_4$ is a square, so are the segments $G'M_1$ and G_0M_3. Once we see that the angle $G'M_1G$ is identical to the angle G_0M_3G and hence equal to $\tan^{-1}(1/\xi)$, all the various segment lengths specified in Fig. 2 can be determined as a function of ξ. Note that so far we have worked only with the given value of ξ and not specified the parameter d.

We now turn to the parameter d. On joining M to M_7, and extending the line backwards so that it intersects the 45° line at a point to be designated as O. This furnishes another coordinate system, and we obtain the instance of the model depicted in Fig. 1 with M_7 labeled there as G_1, and the line MG_1 to be part of the OD line. We have d_ξ to be none other than the slope of the line MM_7. On extending the MV line so that it intersects the vertical at a point to be designated V, we obtain a_ξ.

Alternatively, on joining M to M_6, and extending the line backwards so that it intersects the 45° line at O and furnishes another coordinate system, we obtain the instance of the 2-sector RSS model depicted in Fig. 3, again with the line MM_6 part of the OD line. Again, we have d_ξ as the slope of the line MM_8 line, and can compute a_ξ for this second instance of the RSS model.

It is of some interest, and emphasizes the synthetic nature of our construction, that the value of a_ξ corresponding to the two instances being considered in this essay, are also given by the angles $\angle M_8G''M_1$ and $\angle M_8G'M_1$, respectively, as indicated in Fig. 2.

Finally, on joining M to M_8, and extending the line backwards so that it intersects the 45° line at O and furnishes another coordinate system, we obtain the instance of the 2-sector RSS model with the specification $\xi(1-d)=1$, a specification studied in Khan and Mitra (2005b), one that was shown to yield a continuum of optimal 4-period cycles for all $\rho<(1/\xi)$.

The reader has presumably noted that our discussion has been silent on 2-period cycles and the specification of the discount factor ρ. In the case of 4-period cycles, the third case considered above, the check-map VMD (of say Figs. 1 and 3) has been shown to be the optimal policy function for all $\rho<(1/\xi)$, and hence any value of ρ_ξ less than $(1/\xi)$ would serve. We turn to the other two cases below: first in showing that for either, any value of ρ_ξ less than the corresponding a_ξ would serve, and then, at a second pass, strengthening this to the value $(1/\xi)$ for the first case of convergence to the golden-rule stock in 3 periods. The case of the optimality of the 2-period cycle starting at the point C in Figs. 1 and 3 then also follows.

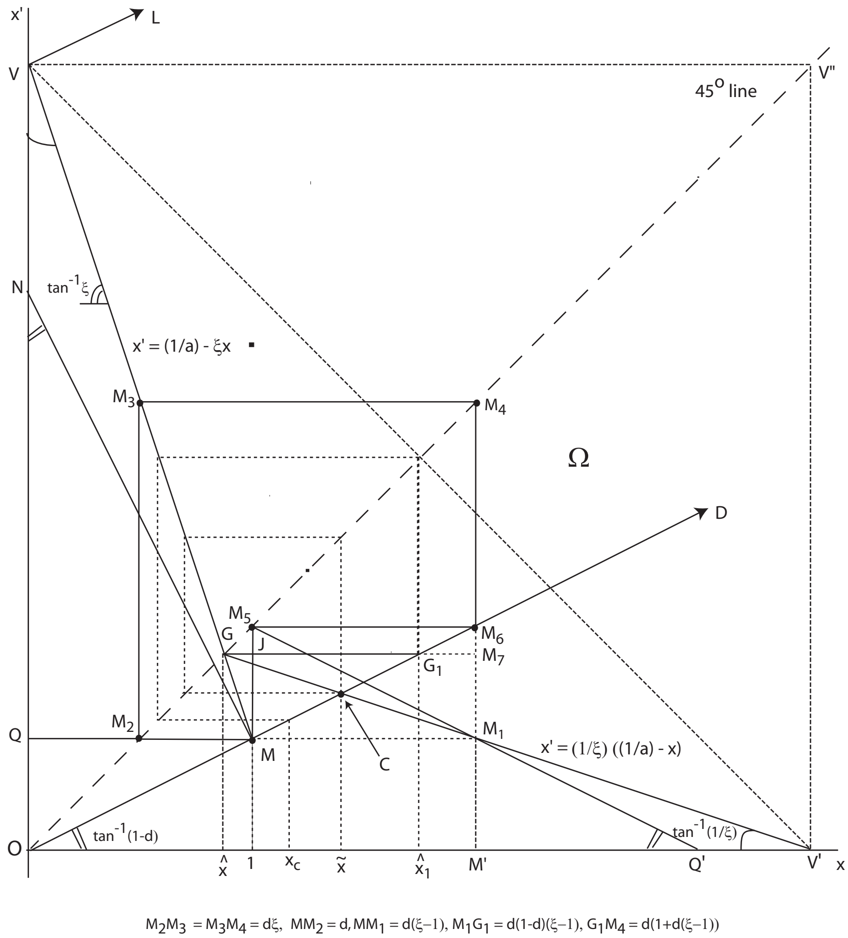

Fig. 3. Basic geometrical benchmarks of the 2-sector RSS model: The case $a(\xi^2 - \xi + 1) = (\xi - 1)$ or $(\xi - 1)(1 - d) = 1$.

The demonstration of the claim is now complete, and before we turn to a brief consideration of the constructions proposed in the antecedent literature,[38] we alert the reader to the fact that in Proposition 2 above, any $\xi > 1$ can be admitted if the demand is for a RSS parametrization that exhibits only 2-period cycles.[39]

3.2. Other constructions

The invocation of the Sonnenschein–Mantel–Debreu insight that a finite exchange economy can be constructed so that its excess demand function is any arbitrarily given continuous function satisfying Walras' law in the context comparative-static properties of multi-sectoral capital theory has already been referred to above. Even though not explicitly mentioned in the context of optimal dynamics in the pioneering 1986 papers of Boldrin, Denekere and Pelikan,[40] it is clearly a sub-text of these investigations. The problem is therein clearly posed: construct a 2-sector technology, a felicity function, a discount factor and an initial capital stock such that an optimal program starting from this capital stock is topologically chaotic.[41] Boldrin and Montrucchio (1986a) work in the context of two settings: (i) the Ramsey–Cass–Koopmans aggregative growth model with the felicity function depending on the capital stock, and (ii) a 2-sector model dual to the one considered here in that it is machines that are produced using both machines and labor, and the consumption-good is produced using capital alone.[42] Whereas no explicit construction is given, the importance of the non-monotonicity of the optimal policy function is identified, and it is clear that the "arguments are enough to conclude that the dynamics induced

[38] The subsequent subsection is prompted by questions from the Editor.

[39] We return to an elaboration of this observation in Section 6 below.

[40] See Boldrin and Montrucchio (1986a,b) and Denekere and Pelikan (1986).

[41] As mentioned in the introduction, towards this end, the authors present a specific parameterized setting so that the optimal policy function is given by the logistic map. The reader is also referred to Proposition 2 above in this context.

[42] See Footnote (14) above in this connection.

by iterating θ (and the τ_ρ for low values of ρ) can be very rich and even chaotic, depending on the relative values of the parameters."[43] Boldrin and Montrucchio (1986b); Denekere and Pelikan (1986) can be seen as (contemporaneous) refinements in the context of the standard 2-sector Srinivasan–Uzawa model of optimal growth whereby sufficient conditions are adduced so that the optimal policy function is a logistic curve. The importance of factor intensity reversals and the negativity of the second cross-derivative of the felicity function is emphasized, with specific bounds being furnished for the discount factor.[44]

It is in the work of Nishimura–Yano that the emphasis shifts from a pursuit of this substantive point to "simple, systematic constructions," and specifically to those involving the tent-map rather than the logistic function. Thus Nishimura and Yano (1994) write, "Our idea is to focus on cyclical optimal paths of period 3 that alternate to be on the boundary of feasible set for two periods and to be in its interior for one period. The characterization of such an optimal path is not obvious and requires a rigorous mathematical argument, to which we devote a large part of this study." This is followed by the authors in 1995 with a shift to ergodic rather than topological chaos. We invite the reader to compare the construction presented above to that in Nishimura and Yano (1994, 1995): the point is that by working in the 2-sector RSS model rather than the full Uzawa–Srinivasan two sector model, we can reduce everything to the single parameter parameter ξ rather than the manipulation three parameters in the fuller model.[45] In this sense, the relevance of the parsimonious RSS model to the issue at hand seems to have been missed.

It is Nishimura and Yano (2000) that refines the authors' earlier (and other) work and summarizes the state-of-the-art of the subject relative to which the marginal contribution of this essay ought to evaluated. They write, "We provide a constructive method [and] the model is very simple"; and show that neither the reversal of factor intensities nor the negativity of the cross-derivatives of the felicity function are a substantively relevant consideration, and that the optimal policy function is expansive and unimodal. Furthermore, rather than the construction of models that show the existence (or non-existence!) of optimal (topological or ergodic) chaotic trajectories, the authors shift the emphasis to the lower bounds on the discount factors and on parametrizations that generate a dynamic of particular periodicity.[46] consider specific constructions for cycles of period 3, 12 and 24, and "demonstrate that every time the order of periodicity is doubled (starting from 3 to 24), the discount factor with which chaotic optimal accumulation may appear is approximately "square-rooted." We have already seen in Proposition 2 that our construction and the underlying argumentation is simpler still and algebraically clean in the sense that it can be executed with an unmarked ruler and compass. Furthermore, as we shall see in Section 7 below, the construction we provide allows the consideration of cycles of arbitrary periodicity and turbulence "in one go," so to speak, and that this "square-rootedness" is simply a technical artifact of the fuller 2-sector setting.

We now turn to the implications of the construction for optimally chaotic dynamics in the 2-sector RSS model.

4. Existence of chaotic trajectories

The optimality argument presented by Khan and Mitra (2005a) revolves crucially on the claim that any program originating from the unit-capital stock in a way that only the consumption sector is active in the initial period, and there is a full employment of resources in the next period, if not in itself optimal, cannot have a larger capital stock in the successive period than the optimal program; see their Corollary 2. This claim is presented in the context of the existence of a unique optimal program from every initial stock if $\rho < a$. All this is valid for the 2-sector RSS model in general, and as such transcends the special cases to be considered here, and in this section we record this antecedent result.[47]

Proposition 3. *If $\rho < a$, the optimal policy correspondence for the 2-sector RSS economy E is a continuous policy function on the state space $[0, 1/ad]$. Furthermore, if $\{x'(t), y'(t)\}$ is an optimal program from 1, then for every program $\{x(t), y(t)\}$ with first two terms $\{(1, 1), (1-d, 1-d)\}$, we must have $x(2) \le x'(2)$.*

The remainder of this section is divided into two subsections, one for each of the two cases that are at issue: the restriction specified as (1) on the one hand, and that specified as (2) on the other. For each, rather than an entire policy function, we show only the optimality of three points on it: the points M, M_3 and M_7 in Fig. 1, and the M, M_3 and M_6 in Fig. 3. The result in the second subsection is new.

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[43] See Boldrin and Montrucchio (1986a; p. 700). In this quote, τ_ρ is the optimal policy function depending on the discount factor ρ, and θ is the basic dynamic equation of the model.

[44] For these claims, see Figs. 1 and 2 in Denekere and Pelikan (1986) (who make explicit Sharkovsky's theorem and notions of ergodic chaos, both considered in below), and Theorem 2 in Boldrin and Montrucchio (1986b). The reader is also invited to compare the transparent simplicity of the bound furnished in Proposition 2 above to that in this latter theorem; the appeal to Rockafellar's notion of α-concavity in their construction should also perhaps be noted in such a comparison.

[45] See Appendix A, which relies on the proof of their Lemma presented in Appendix A in Nishimura and Yano (1995). Such argumentation can be totally bypassed in the setting that we present here.

[46] The authors write, "We provide an example in which optimal paths are ergodically chaotic in the case in which future utilities are discounted by about 11% (or, to be precise, for the case of $\rho = 0.886$). In the existing examples of ergodically chaotic optimal accumulation, in contrast discount factors are set around $\rho = 0.01$. See the discount factors implied by Proposition 2 above. We may observe here that parametric restrictions that are necessary and sufficient for topological and ergodic chaos are derived in ongoing work by Khan and Mitra.

[47] It bears emphasis that the sufficient condition involves a restriction on the discount factor rather than an appeal to the strict concavity of the reduced-form utility function, as in Mitra (2000a; p. 53). The latter property is not available to us here.

4.1. Optimal 3-period convergence

The distinguishing characteristic of the particular instance of the 2-sector RSS model considered earlier consists in the fact that a full employment of resources in the first-two periods leads, with zero investment, to convergence to the modified golden-rule stock in the third period. This is pictured in Fig. 1, and the first equality in (1) is an algebraic representation of this fact.

We can now claim that the program $\{x^{\sigma}(t), y^{\sigma}(t)\}$ constituted by the successive plans M, M_3, G_1 and G in Fig. 1 is an optimal program. Suppose that this was not the case. We are guaranteed that there is an optimal program from the unit capital stock (see Mitra (2000a), for example). If such a program starts at a plan above J on the vertical through M and J, the program constituted by the successive plans J and G gives lower aggregate value-loss and thereby furnishes a contradiction. Thus suppose that the first two plans of the optimal program are given by α and β, where α lies in the segment MJ, and β lies on the vertical through the point at which the horizontal through α intersects the 45°-line. (These plans, being arbitrary are not shown in Fig. 2; we encourage the reader to put them in for herself.) Given Proposition 1, certainly β lies above the horizontal M_3M_4. But then the program $\{x^{\sigma}(t), y^{\sigma}(t)\}$ gives a lower aggregate value-loss and furnishes a contradiction.

We can now appeal to the proof of Theorem 1 in Khan and Mitra (2005a,b) to state

Theorem 1. *Under the restriction specified as (1) there exists an optimal topologically chaotic program for the 2-sector RSS model with $\rho < a$.*

4.2. Optimal 3-period cycles

The distinguishing characteristic of the particular instance of the 2-sector RSS model that we present here consists in the fact that a full employment of resources in the first-two periods leads, with zero investment, to a unit capital stock in the third period. This is pictured in Fig. 3, and the first equality in (2) is an algebraic representation of this fact. This simplifies to the second equality in (2) asserting the fact that the line NM, (VN being a segment of unit length) is perpendicular to the OD line.

We can now claim that the program $\{x^{\pi}(t), x^{\pi}(t+1)\}$ constituted by the successive plans M, M_3, M_6 and M in Fig. 2 is an optimal program. Suppose that this was not the case. Again, we are guaranteed that there is an optimal program from the unit capital stock, and such a program cannot start at a plan above J on the vertical through M and J. Thus, as before, suppose that the first two plans of the optimal program are given by α and β, and successive plans labeled β_i, $i = 1, 2, \ldots$. (As in the subsection above, these plans, being arbitrary, are not shown in Fig. 3, and we encourage the reader to put them in for herself.) Certainly, Proposition 1 guarantees that the plan β lies on or above M_3M_4 on the vertical through the intersection of the 45°-line with the horizontal through α. Now let the vertical through M_6 intersect the horizontal at V at γ. If β_1 lies below or equal to γ on the vertical through the intersection of the 45°-line with the horizontal through β, consider the program constituted by the successive plans M, M_3, M_5, β_i, $i \geq 1$. On the other hand, if β_1 lies above γ, consider the program constituted by the successive plans M, M_3, M_6, J and G. In either case, the constructed program gives a lower aggregate value-loss than the alleged optimal program, and furnishes a contradiction.

We can now state

Theorem 2. *Under the restriction specified as (2) there exists an optimal topologically chaotic program for the 2-sector RSS model with $\rho < a$.*

Proof. For the function h obtained from Corollary 1, note that $h^3(x^*) = x^* > h(x^*) > h^2(x^*)$, where $x^* = (1/a) + (1-d)\xi = (d/a) - (1-d)^2$. We can now appeal to the Li–Yorke theorem (see for example Mitra (2001; Proposition 2.1) or Li and Yorke (1975)) to complete the proof. □

5. Existence of chaotic trajectories: a lower bound

Once the restrictions (1) and (2) are pictorially represented as Figs. 1 and 3, a natural question arises as to the possibility of a self-contained geometric proof of the existence of chaotic trajectories. Furthermore, since it is now known that the optimal policy correspondence is a function for all values of $\rho < 1/\xi$ in all cases of the 2-sector RSS model in which the slope of the OD line is less that of the MM_7 line, the further question is raised as to whether one generalize Theorems 1 and 2 above to allow ρ to be larger than or equal to a. We turn to this in this below.

5.1. Optimal 3-period convergence

As we have seen above, in an optimal program no plan above the point J will be chosen from a unit capital stock by virtue of the slope of the constant value-loss lines. Thus suppose that there is an optimal program whose element is given by a plan α lying in the open interval JM, and successive plans labeled β_i, $i = 1, 2, \ldots$.[48] Now let the vertical through β_1 intersect VV'' at

[48] As in the subsection above, these plans, being arbitrary, are not shown in Fig. 1, and we encourage the reader to put them in for herself.

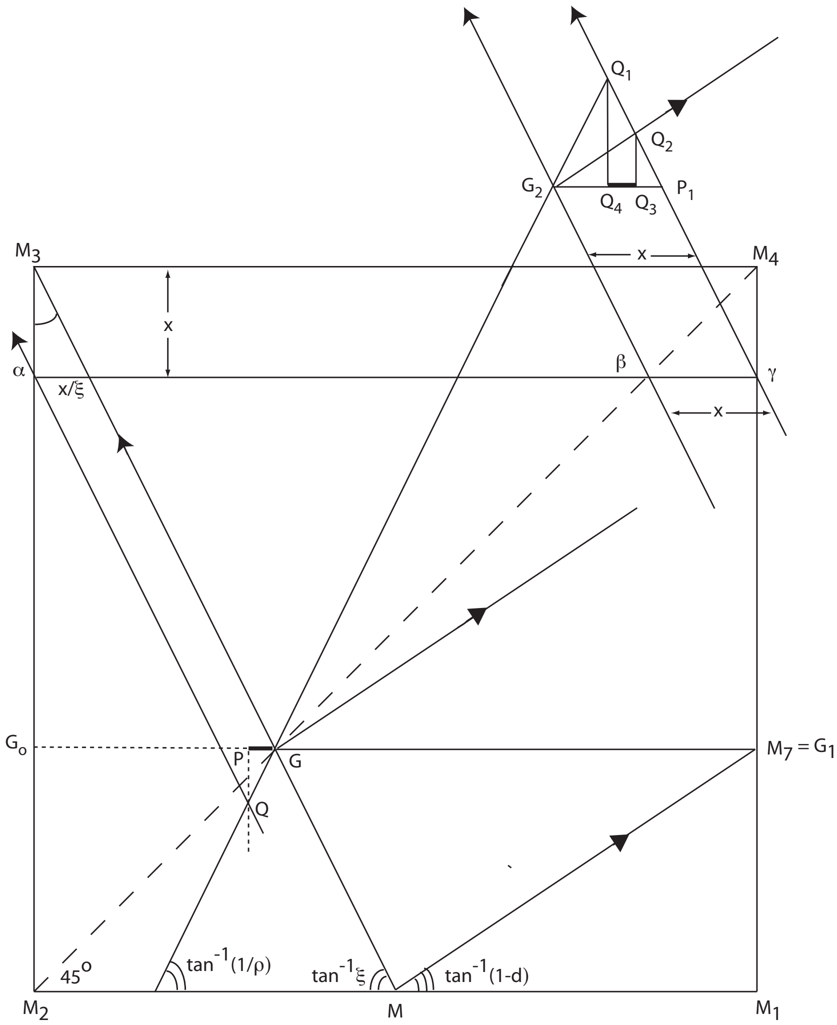

Fig. 4. (a) Computation of value-loss differentials: The case $(\xi - (1/\xi))(1 - d) = 1$. (b) A simple property of an acute-angled triangle. (c) Magnification of value-loss differentials.

γ. If β_1 lies below or equal to γ, consider a program that consists of a plan given by the plan M, followed by the subsequent plans β_i, $i = 1, 2, \ldots$. Such a program is feasible, and has a lower aggregate value-loss, and thereby contradicts the alleged optimality of the given program. Thus suppose that $\beta_1 > \gamma$. Since an optimal program is good, there will exist an index $i' \neq 1$ at which $\beta_{i'} \leq \gamma$. Then simply consider a program that is identical to the given program for the first plan α and all plans equal to and succeeding $\beta_{i'}$, but with the other plans being γ. Such a program is again feasible and contradicts the alleged optimality of the original program. Thus, on the optimal program, the first plan must be either J or M. Let us suppose it to be M.

In the "trapping square" magnified as Fig. 4a, it is clear that any plan below G_o on the vertical through M_2, M_3, cannot be part of an optimal program. We have also seen above that any plan on this vertical above M_3 cannot be part of an optimal program. Thus suppose the plan α (in the interval $[G_o, M_3)$ that does not include M_3) is part of the optimal program, and that it is followed by β. If β lies below GG_1, consider a program that is constituted by M_3, G_1 and G. Such a program is feasible, and has a lower aggregate value-loss, and thereby contradicts the alleged optimality, of the given program. Thus suppose that β lies on or above GG_1. In this case, the feasible program that consists of M_3, and is identical to the given program from there on, has a lower aggregate value-loss, and again contradicts the alleged optimality. This claim involves a trade-off between value-loss due to unemployment and that due to excess-capacity of capital; the bolder segments indicated in Fig. 4a and c. This computation is independent of the discount factor ρ and relies on a simple property of an acute-angled triangle presented in Fig. 4b. We present its outline in Fig. 4c, and relegate the precise details to Appendix A.[49]

[49] We suggest that the interested reader return to this paragraph after working through the Appendix A.

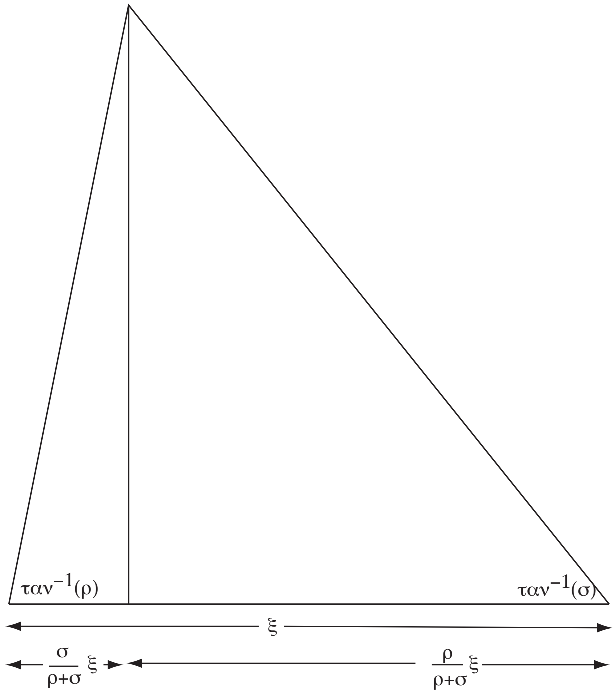

Fig. 4. (*Continued*)

All that remains is to show that for $\rho < (1/\xi)$, the plan J can never be part of an optimal program. Towards this end,[50] consider two alternative programs: the first where the planner moves to the golden-rule stock and stays there (the straight-down-the-turnpike path), and the second, the path that returns to the initial capital stock after three periods. In Fig. 1, simply observe that by virtue of the similarity of the triangles, GJM and GG_1M_1,

$$\delta^{\rho}(1, \hat{x}) = \rho^2 \delta^{\rho}(\hat{x}_1, \hat{x}) \Rightarrow \rho^2 = \frac{\delta^{\rho}(1, \hat{x})}{\delta^{\rho}(\hat{x}_1, \hat{x})} = \frac{GJ}{G_1G} = \frac{GJ}{JM}\frac{G_1M_1}{G_1G} = \frac{1}{\xi^2}.$$

But is it easy to check that as ρ falls, and the benchmark line with slope $(1/\rho)$ becomes steeper, the value at M is lower than at J.

The argument is complete.

5.2. *Optimal 3-period cycles: an example*

A natural question arises as to whether the optimality of the 3-period cycle considered in the specific instance (2) and in Fig. 3 also extends to situations when $\rho \geq a$. We answer this question in the negative by considering the specific numerical case of $\xi = 3$ and therefore $d = 1/2$ and $a = 2/7$. A quick computation reveals that at the discount factor (1/3), the discounted value of consumption at the straight-down-the-turnpike program is higher than at the 3-period cycle (1.3304 compared to 1.3269). We leave it to the reader to provide analogous calculations in terms of the aggregate value-loss of the two programs.

5.3. *2-Period cycles*

So far we have focussed our entire attention on optimal 3-period convergence to the golden-rule stock and to 3-period cycles. It has already been established in Khan and Mitra (2005b) that the 2-period cycle is given by the intersection of the *OD* and the dual *MV* lines, and this observation utilized in the synthetic claim underlying the construction presented in Section 3 above. We leave it to the reader to show that in Fig. 4a, the ratio of the segments GC_{12} and GC_{11}, representing the aggregate value-loss of the 2-period cycle to that of the straight-down-the-turnpike program respectively, is given by ξ^2, and therefore equal to $(1/\rho^2)$.

An identical claim can be made for the second example, and verified through the geometry presented in Fig. 3.

[50] This part of the argument is a simple application of the technical apparatus presented in Khan and Mitra (2005b).

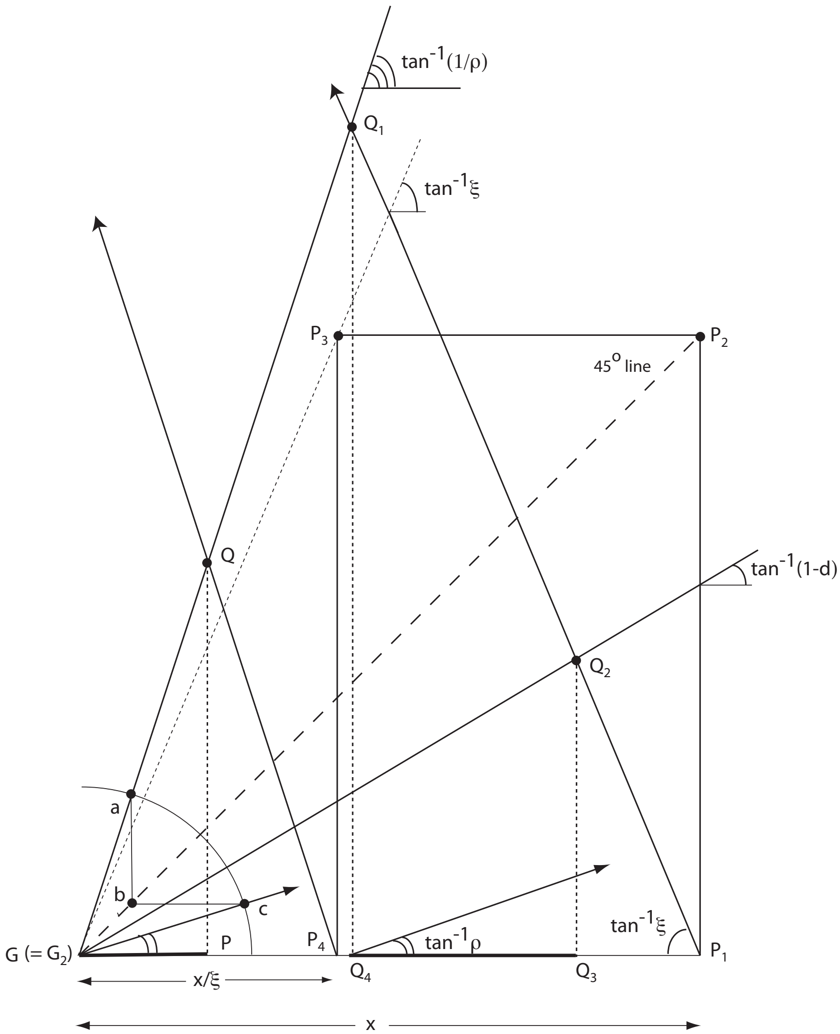

Fig. 4. (*Continued*).

6. The construction: additional considerations

It was mentioned in the introduction that the existence of an instance of the 2-sector RSS model satisfying (1) is established through an application of the intermediate-value theorem in Khan and Mitra (2005b). Here we observe that the construction of Section 3, and its focus on the trapping square $M_1M_2M_3M_4$ allows a simple constructive proof of existence. Such a construction is presented in Fig. 5 and involves bisecting the unit interval M_2M_1, say at M, and completing the unit square $M_1M_2M_3M_4$. The MV line of slope 2 is given by MM_3, the golden-rule stock G by its intersection with the 45° line, the marker M_7 a consequence of G, the line GM_1 of slope 1/2 the dual MV line, the line MM_7 the OD line with slope 1/3, and finally, by joining G to M_o determined by the bisection of GG_1, we obtain the benchmark line of slope $1/\rho = 2$. In Fig. 5, the value-losses at $\rho = 1/2$ are indicated by the bold segments.[51]

The feasibility of the construction revolves around the fact that the line MM_7 is flatter than the 45° line, which is to say that it is less than unity. It is easy to show in Fig. 5 that the line MM_6 is parallel to the 45° line, and thereby highlights an aspect of our construction not emphasized in Section 3. This is simply the fact that its viability as a synthetic construction hinges crucially on the fact that each of the benchmark points M_6, M_7 and M_8 are laid out in a way that the resulting joins to M are not steeper than the 45° line. Put more technically, the given $\xi > 1$ must be such that the resulting $d_\xi \geq 0$. Thus in Fig. 5, a $\xi < 2$, would lead the corresponding MM_6 to intersect with the 45° line.

[51] As we have already seen, Fig. 4c allows the derivation of a line with slope $(1/\rho)$, and it can be used to show that the aggregate value-loss of one trajectory is less than the other. Note also that the property of the intersection points of the value-loss lines at G_o and G_1 and the GM_o line portrayed in Fig. 5 is a consequence of $\rho = \xi$ and not of $\xi = 2$.

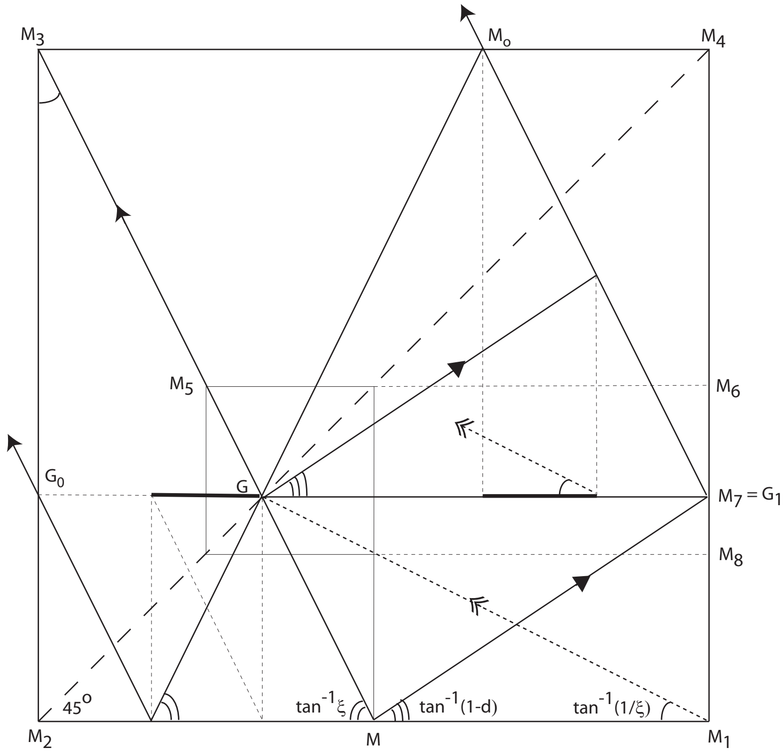

Fig. 5. The case $(\xi - (1/\xi))(1 - d) = 1$ in the specific instance $\xi = 2$, $d = 2/3$.

But this leads to the formulation of an interesting question that is provoked by the construction presented in this essay. Since ξ serves as a sufficient statistic for the construction of the 2-sector RSS model for a pre-chosen dynamic, the natural question arises as to the smallest value of ξ that serves to generate the chosen dynamic. And as we have seen above, we obtain the value 2 in connection with a 3-period cycle. This meshes with the fact that (2) gives d to be zero when ξ equals two. We now realize that a unit value of ξ is the minimum for 2-period cycles, and any number greater than 2, for 4-period cycles, observations that conform to the findings in Khan and Mitra (2006a). In this context, we may also note that the construction presented in this essay allows transparent proof of the observation that the non-existence of a 2-period cycle is both necessary *and* sufficient for the global asymptotic convergence of optimal programs to the golden-rule stock. We leave it to the reader to rework Fig. 1 to show that (i) a unit value of ξ leads to the collinearity of the lines M_3G and GM_1, which is to say that the point G is the center of the square. This leads to a continuum of 2-period cycles. For values of ξ within the unit interval, and with the slope of the line MM_3 less than 45°, we need to extend the 45° line "backwards" to intersect the downwards vertical drawn from M_3, and thereby complete the square with a displaced origin O. The resulting RSS model has the check-map as its optimal policy function for all discount factors, and thereby leads to global asymptotic convergence. There is of course a dual-value line whose intersection with the corresponding OD line gives a 2-period cycle, but this is no longer on the graph of the optimal policy function.[52]

The smallest value of ξ that leads to a program that converges to the golden-rule stock from unity is especially interesting. It is the solution of the following quadratic equation

$$\xi^2 - \xi - 1 = 0, \tag{7}$$

and furnishes the root $\xi = (1 + \sqrt{5})/2$. And since the highest discount factor under which such a program is optimal is $\xi = (\sqrt{5} - 1)/2$, this rather intimately to the golden number.[53] But this leads directly to ask for the smallest value of ξ, and the least upper bound of the discount factor, under which there is convergence in 4 periods to the golden-rule stock from the unit capital stock. This question requires a revisit of the construction presented in Section 3.

[52] This is admittedly only a sketch of the argument, and the reader is invited to draw on the relevant results in Khan and Mitra (2006a) to complete it. Note that the basic observation of the necessity *and* sufficiency is due to Coppel (1955), and exploited by Mitra (2005) through the use of the Euler–Lagrange conditions, a technique relying on the differentiability of the felicity function at the golden-rule stock that is not available to us here.

[53] See Footnote 3 in Mitra (1996), and also the other references furnished in Footnote (24). It bears emphasis that we are being led to this number by considerations, perhaps parallel, but certainly not the same as Mitra (1996); Nishimura and Yano (1996).

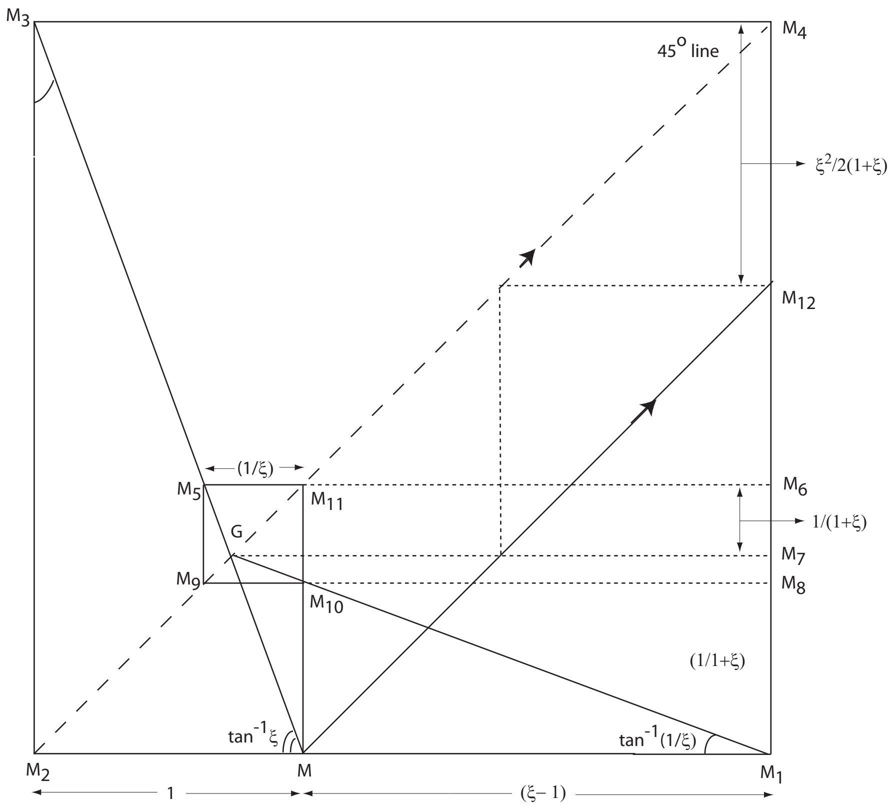

Fig. 6. Convergence to the golden-rule stock in 3 periods.

Now, if ξ is steep enough, in addition to the benchmarks M_6, M_7, M_8, we cam mark out the benchmark M_{12}. If MM_{12} is not steeper than the 45° line, we can use it to figure out the 2-sector RSS model (d_ξ, a_ξ, ρ_ξ), $\rho_\xi < a$, that yields convergence to the golden-rule stock in 4 periods. The smallest value of ξ that leads to such a program can now be computed. If we denote the length of the segment M_4M_{12} by x, we obtain

$$2x - \frac{1}{(1+\xi)} = \xi - 1 \Rightarrow x = \frac{\xi^2}{2(1+\xi)}.$$

But we can now obtain the following generalization of the quadratic Eq. (7) presented above:

$$\xi^2 - 2\xi - 2 = 0 \Rightarrow \frac{2}{x} = \frac{x}{1+x}. \tag{8}$$

But now the relevant generalization of Eq. (7) is straightforward; the smallest value of ξ that yields convergence to the golden-rule stock in $(k+2)$ periods with $k = 1, \cdots$, is given by the equation

$$\xi^2 - k\xi - k = 0 \Rightarrow \frac{k}{x} = \frac{x}{1+x}. \tag{9}$$

Thus, rather than finding an extension of the unit interval such that the extension bears to the extended interval exactly what the unit interval bears to it, as in Eq. (7) for the golden number, we substitute the phrase"k times", $k = 1, \ldots$, for "exactly." On all this, see the geometric substantiation in Fig. 6.

7. The construction: a further elaboration

There is an aspect to the construction presented in Section 3, and used in Sections 4–6, that is not fully satisfying. This has to do with a theorem due to Sharkovsky that we recall for reader's convenience.[54]

[54] Several expositions of this theorem are by now available in the economic literature; see for example Chapter 1 in Majumdar et al. (2000); also Kelsey (1988); Rosser (2000), the *JEBO* symposia edited by Medio (1987); Day and Eliasson (1991) and Chapter 3 in Majumdar (2009). For the mathematical literature, see the references in Footnote (9). In this exposition, we follow Elaydi (1996) who calls the result "spectacular".

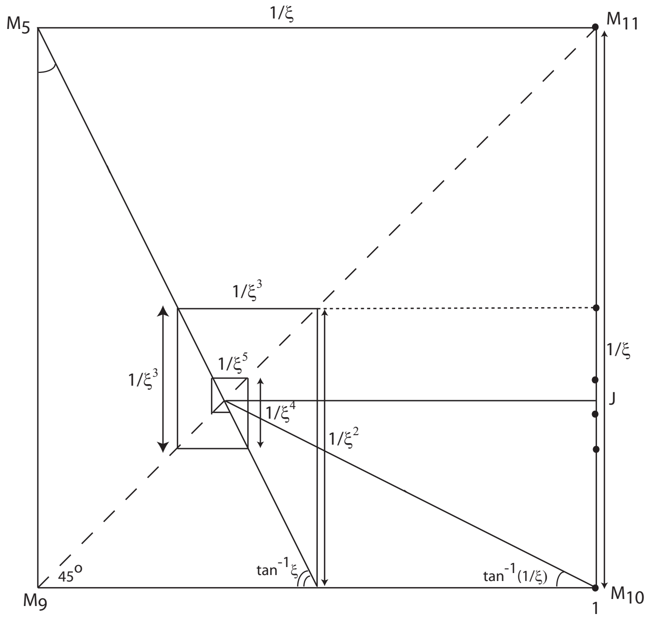

Fig. 7. Elaboration of the Construction for $\xi > 1$.

Consider the following ordering of the set of real numbers due to Sharkovsky (1964):

$$\underbrace{3 \lhd 5 \lhd 7 \lhd \cdots}_{\text{odd integers}} \underbrace{2 \times 3 \lhd 2 \times 5 \lhd 2 \times 7 \lhd \cdots}_{\text{twice odd integers}} \underbrace{2^2 \times 3 \lhd 2^2 \times 5 \lhd 2^2 \times 7 \lhd \cdots}_{\text{twice squared odd integers}} \underbrace{2^3 \times 3 \lhd 2^3 \times 5 \lhd 2^3 \times 7 \lhd \cdots}_{\text{twice cubed odd integers}} \cdots\cdots\cdots$$

$$2^4 \lhd 2^3 \lhd 2^2 \lhd 2 \lhd 1$$

Theorem (Sharkovsky, 1964). *Let f be a continuous map from the unit interval to itself. If $k \lhd r$, which is to say that k appears before r in the Sharkovsky ordering, and f has a point of period k, then f must have a point of period r.*

An obvious corollary of the theorem is that a map that exhibits a 3-period cycle exhibits cycles of all periods. Now consider the construction as presented in Section 3 and summarized in Fig. 2 with a value of ξ that leads to a parametrization of the 2-sector RSS model exhibiting an optimal 3-period cycle. This is the model where $(1-d)$ is given by the slope of the line joining M and M_6, and the value of ρ to be chosen as less that $a = (1/(\xi + (1-d))$, as dictated by the principal result in Khan and Mitra (2007b,c) that the check-map (the M_3M and MM_6 segments in particular) is the optimal policy function for all discount factors below a. As exhibited in Fig. 2 (or in Fig. 6), the value of ξ has enough "slack" in it, which is to say, is large enough, that a 3-period cycle is optimal. Another way of making this observation is to say that the line MM_6 is "flatter" than the 45° line. By way of contrast, in Fig. 5, this is not so. Now Sharkovsky's theorem assures us that all the values of ξ that yield a three period cycle yield cycles of all periods. Two questions then arise. First, how is one to find cycles of other periods in a particular instance of the 2-sector RSS model? This is straightforward but more difficult question. Second, given a value of $\xi > 1$ that yields an instance of a 2-sector RSS model exhibiting an optimal 3-period cycle, how is one to find instances of the 2-sector RSS model with the *same* ξ that yields a cycle of a given period? we shall not have anything to say on the first question, but a fully satisfactory answer to the second question can be given.[55]

Consider Fig. 3, and focus on the square $M_5M_9M_{10}M_{11}$. The solution is simply to continue with smaller squares of sizes $1/\xi^3, 1/\xi^5, \ldots$ as in Fig. 7 and mark out the points above and below J. By joining these points to the points to the M, instances of the 2-sector RSS model exhibiting cycles of arbitrary order come to light. It should be clear to the reader that for odd numbered cycles, the relevant points lie above J in Fig. 7, and for even numbered cycles, below it. To take a fancy, if not fanciful concretization, for a given $\xi > 1$ that exhibits a 3-period cycle, one can construct versions of the model for cycles corresponding to all the Catalan numbers.[56]

[55] These considerations have already been alluded to in the second paragraph of Section 6.2 above.

[56] This inquiry is inspired by what the authors see an analogy in Figure 17.2 in Mas-Colell (1989). For a discussion of Catalan numbers, see Borwein and Bailey (2008). It would be pedantic to present formulae and equivalences but we recall an initial list for the reader: 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, $\cdots$.

Next, we turn to two refinements of Sharkovsky's theorem. The first concerns Nathanson's (1977) generalization whereby a map on the interval is chaotic in the Li–Yorke sense even if it exhibits cycles of 5 or 7 periods. Fig. 7 shows that the right arm of the check-map of the 2-sector RSS version corresponding to these two periodicities also lies *above* the point *J*, and as such, Nathanson's extension does not enable us to show Li–Yorke chaos for what are termed borderline and inside cases in Khan and Mitra (2006b).

The second refinement draws on the Block and Coppel results on turbulence which were used by Khan and Mitra (2005b) to show that the second iterate of the check-map corresponding to the borderline case has positive entropy, and hence is turbulent, and hence the function itself is chaotic.[57] For this, we need the notions of *turbulence* and *strict turbulence*. Following Block and Coppel (1991), we shall say that an arbitrary continuous map *f* of the interval to itself is *turbulent* if there exist compact subintervals *J*, *K* with at most one common point such that

$$J \cup K \subseteq f(J) \cap f(K).$$

It is said to be *strictly turbulent* if the subintervals *J*, *K* can be chosen disjoint. Now let I\kern -1.7pt K denote the set of all continuous maps $f: I \to I$, I the unit interval, which are chaotic, and $\mathbb{P}_n$, $\mathfrak{T}_n$, $\mathfrak{S}_n$ be the respective set of all continuous maps $f: I \to I$ such that *f* has a periodic point of period *n*, f^n is turbulent, and f^n is strictly turbulent. Then the *Sharkovsky stratification*

$$\mathbb{P}_3 \subseteq \mathbb{P}_5 \subseteq \mathbb{P}_7 \subseteq \cdots \subseteq \mathbb{P}_6 \subseteq \mathbb{P}_{10} \subseteq \cdots \subseteq \mathbb{P}_{12} \subseteq \mathbb{P}_{20} \subseteq \cdots \subseteq \mathbb{P}_8 \subseteq \mathbb{P}_4 \subseteq \mathbb{P}_2 \subseteq \mathbb{P}_1$$

admits the following refinement by the *turbulence stratification*

$$\mathfrak{S}_1 \subseteq \mathfrak{T}_1 \subseteq \mathbb{P}_3 \subseteq \mathbb{P}_5 \subseteq \mathbb{P}_7 \cdots \mathfrak{S}_2 \subseteq \mathfrak{T}_2 \subseteq \mathbb{P}_6 \subseteq \mathbb{P}_{10} \subseteq \mathbb{P}_{14} \cdots \mathfrak{S}_4 \subseteq \mathfrak{T}_4 \subseteq \mathbb{P}_{12} \subseteq \mathbb{P}_{20} \subseteq \mathbb{P}_{28} \cdots\cdots\cdots \mathbf{K} \subseteq \cdots \subseteq \mathbb{P}_8 \subseteq \mathbb{P}_4 \subseteq \mathbb{P}_2 \subseteq \mathbb{P}_1$$

where $\mathbf{K} = \cup_{n>0} \mathfrak{T}_n$. This refinement underscores the importance of the construction presented in this essay. In the context of Fig. 7, the point *J* represents the point M_7 in Fig. 6, and we know that the second iterate of the optimal policy resulting from this 2-sector RSS parametrization is turbulent. Then, by the turbulence stratification presented above, we know that it exhibits cycles of all even periods. On the other hand, by joining *M* to the point M_6, we obtain a parametrization for which f^3 is both strictly turbulent and turbulent. Furthermore, if instead of M_6, we join *M* to any of the points demarcated above *J* in Fig. 7, f^n is both strictly turbulent and turbulent, where *n* is the periodicity corresponding to the chosen point.[58]

A final remark in this section concerns Elaydi's (1996) converse to Sharkovsky's theorem. It is of some interest to see how this translates into the problem being studied here. For a $\xi > 1$ furnishing a particular instance of the 2-sector RSS model that yields an optimum cycle of period *k*, and no higher, it means that the corresponding *OD* line (see Figs. 1 and 3) for a cycle of period $(k+1)$ has a slope equal to or higher than that of the 45° line. Thus, the particular value of ξ in Fig. 5 will not allow an optimal cycle of higher order than 6, which is more evidently expressed as (2×3) in the statement of Sharkovsky's theorem. In other words, for any point higher than M_7 on the M_1M_4 vertical, the corresponding line connecting that particular point to *M* has a slope steeper than that of the 45° line. It is also of interest that Elaydi's construction has a part corresponding to the check-map.

8. Conclusion

We begin this conclusion with May's (1976) valorization of simplicity and Saari's (1995) resigned acceptance of it in the economic sciences: rather than an observation about the 'real world', both are simply revealing a preference for models with explicit low-dimensional functional forms as opposed to more sophisticated analytical tools; the tent-map or the logistic function being canonical examples of such forms. Or to put the matter more succinctly, everything else being same, the simpler the tool the better.[59] From this angle, the 2-sector RSS model, despite its tri-parametric simplicity and geometric tractability,[60] continues to offer surprises. As mentioned already, in terms of professional acceptability, it seems to have fallen through the cracks between the Ramsey–Cass–Koopmans model and the full-fledged 2-sector Shinkai–Uzawa–Srinivasan model.[61] Surely, in terms of the complexities of capital theory, it has as much to teach the community as any other construct.

To be sure, this is not an argument against sophisticated analytical tools or empirical calibration. Theorems, most certainly, are not there only for ideological reassurance, and general models such as the one due to Arrow, Debreu and McKenzie for

[57] The relevant results, utilized in Khan and Mitra (2005b) are that the topological entropy of the k^{th} iterate of continuous map from the unit interval to itself is *k* times the topological entropy of the map itself (Proposition 2, p. 191); and that the topological entropy of such a map is greater than or equal to log 2 if the map is turbulent (Corollary 15, p. 200).

[58] See Theorem 14 in Block and Coppel (1991). We also note that the extension to accommodate turbulence for the construction presented in this paper follows the discussion of Chapter 2 in this book. The reader is also referred to Block (1986) for a brief accessible account.

[59] We repeat ourselves here, but perhaps the imperative bears repetition; see Footnote (13), and the text it footnotes.

[60] As the reader, even one innocent of the RSS literature, has by now appreciated, the 2-sector RSS model is fully described by the triple (a, d, ρ), all non-negative real numbers, but with *d* and ρ being bounded by unity. Again, it is the single capital good in the 2-sector RSS model, in contrast to the RSS model, that is responsible for its plane-representation and geometric tractability. Also see Figure 17.4 in Mas-Colell (1989) and Footnote (27), as well as the text it footnotes.

[61] It is of interest that Mas-Colell (1989) does not represent Mas-Colell's début into capital theory: for *afficianados* of the 2-sector model, Mas-Colell and Razin (1973) and Bosch et al. (1975) are important precursors.

the study of economic interaction, or that due to Gale and McKenzie for the study of intertemporal allocation, are, in their contexts, also objects of perfection precisely because of the theorems they afford and bring to light. Their proofs are the lens for viewing their perfection. Marshall's dictum that the mathematics be burnt after it has served its economic purpose, the ladder disregarded once it has been climbed over, reveals more about the Cambridge of its time than imperatives of mathematical modeling of the "real world". Perhaps rather than simplicity versus complexity, the issue has more to do with the existence of examples[62] that illuminate general models by calling attention to their lack of generality, not as an end in itself, but to get at some aspect of what Russell, in the first epigraph to this essay, refers to as 'truth'. This essay then is also an attempt to seek social acceptability for the RSS model as a shared vehicle for the exploration of capital theory (theory of intertemporal allocation, if one prefers) and its commanding theorems.

However, we end this essay by coming down from these rather airy and fanciful methodological heights to the firmer grounding of the 2-sector RSS model. We ask questions directly suggested by our analysis.[63] First, what is the least upper bound of the discount factor under which there are 3-period cycles in the 2-sector RSS model? Second, what is the smallest value of ξ, and the least upper bound of the discount factor, under which there is convergence in k, $k \geq 3$, periods to the golden-rule stock from the unit capital stock in the 2-sector RSS model? The second question could be alternatively phrased as asking for the corresponding generalization of the quadratic Eq. (7) presented above. These questions would lead to a consideration of additional refinements in a revisit of the construction presented in Section 3. And of course the complete delineation of the optimal policy correspondence still remains yet to be fully accomplished.[64] Finally, it is important to bear in mind that the 2-sector RSS model is, in the final analysis, based on a technological specification, and as such, can be used to pursue enquiries for descriptive growth theory analogous to those we have reported for optimum growth theory. And so the third question would be to ask for parameters of the RSS technological specification that lead to specified dynamic patterns in say Keynesian, overlapping generations or neoclassical growth models.[65]

Appendix A.

We turn to the computation of value losses discussed in Section 5.1 and in Fig. 4a. Recall that two programs are being compared, the second differing from the first only in that it involves M_3 and γ instead of α. After the plan γ, the second program simply follows in the footsteps of the first with a lag of one period. However, before considering the computation, we note a subsidiary result and two simple constructions.

The result is exhibited in Fig. 4b, and is a simply property of an acute-angled triangle whose proof we leave as an exercise to the reader. The constructions are exhibited in Fig. 4c. The first construction shows how to derive a line whose slope is the reciprocal of the slope of a given line. Thus, in terms of the line of slope $1/\rho$ in Fig. 4c, use G as the center of a circular arc that marks a point a on the line GQ_1. Drop the perpendicular from a to GP_1, and on using its intersection b with the 45° line, mark out another point c on the arc. Join c to G to obtain the line we are looking for. This is easily seen as a consequence of the fact that the triangles Gab and Gbc are congruent and the $\angle Gab$ is $\tan^{-1}\rho$.

Next, we turn to the construction that shows how to obtain the segment of length x/ξ from a given segment of length x. Use the 45° line and a line of slope ξ to obtain P_4 by following the points P_1, P_2 and P_3. Since GP_1 equals P_1P_2 which in turn equals P_3P_4, we obtain that GP_4 equals $(1/\xi)GP_1$.

We can now use these constructions to show that the first program has a higher value loss than the second. For this we recall that the value-loss of a particular plan is given by the shortfall from the golden-rule utility level plus the loss due to excess capacity; see Equation (9) in Khan and Mitra (2006a; p. 360).[66] In Fig. 4a, there is no excess capacity at plan α but only unemployment, and therefore the value-loss is given by the segment PG. For the plans β or γ, the value-loss is given by the intersection of the corresponding value-loss lines with the extended line through QG minus the relevant excess capacity; see Figs. 1 and 2 in Khan and Mitra, 2006a). Thus, the difference in value-losses between the plans β and γ are given by the differences in excess capacities (the segment G_2Q_3 in Fig. 4a), plus the differences in the shortfalls from golden-rule utilities (the segment G_2Q_4 in Fig. 4a). This reduces to the segment Q_4Q_3. We thus need to compare the segment PG with the segment ρQ_4Q_3.

For this, we magnify the triangles PGQ and $P_1G_2Q_1$ of Fig. 4a and embed them in Fig. 4c. Now the point G should be understood to refer both to the point G as in the triangle PGQ and also as the point G_2 in the triangle $P_1G_2Q_1$.

The computation now applies the result presented in Fig. 4b by using successively s to be ξ, and r to be $(1/\rho)$ and $(1-d)$. Now given x as in Fig. 4a, we focus on the triangle GQP to obtain the first bold segment GP in Fig. 4c to be the value-loss

$$\frac{\xi}{(\xi + (1/\rho))}\frac{x}{\xi} = \frac{\rho x}{1+\rho\xi}.$$

[62] Even a cursory acquaintance with Mas-Colell's papers reveals the importance that he attaches to examples. Perhaps one index of the difference between a mathematician and one who is not is by their proclivity towards examples rather than theorems.

[63] See the reference to an affiliated inquiry being pursued by Khan and Mitra in Footnote (19).

[64] See Khan and Mitra (2007b,c) for an inside/outside categorization, and for progress within the inside case.

[65] As, for example, the models of Mas-Colell and Razin (1973); Bosch et al. (1975). For a recent investigation, see Dudek (2010).

[66] We also take this opportunity to correct an error in Fig. 2 in Khan and Mitra (2006a). The value-loss from the plan F in the figure is given by the difference between $\hat{x}$ and the abscissa of T, rather than the segment $[\bar{x} - \hat{x}]$, as indicated in the figure.

Next, on focussing on the triangles GQ_2P_1 and GQ_1P_1, we obtain the first bold segment in Fig. 4c to be the value-loss

$$\left(\frac{\xi}{\xi+(1-d)}-\frac{\xi}{\xi+(1/\rho)}\right)x=\left(\frac{1-\rho(1-d)}{(\xi+(1-d))(1+\rho\xi)}\right)\xi x.$$

But we can now see that

$$\frac{\rho x}{1+\rho\xi}-\left(\frac{1-\rho(1-d)}{(\xi+(1-d))(1+\rho\xi)}\right)\xi\rho x=\left(\frac{\rho x}{1+\rho\xi}\right)\left(1-\frac{(1-\rho(1-d))\xi}{\xi+(1-d)}\right)=\left(\frac{\rho x}{1+\rho\xi}\right)\left(\frac{(1-d)(1+\rho\xi)}{\xi+(1-d)}\right)>0.$$

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