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On the Mitra–Wan forestry model: A unified analysis ☆

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Abstract

We present a substantive and far-reaching generalization of the principal results in the economics of forestry, as formalized by Mitra and Wan (1986). Rather than a polarized dichotomy of linear and strictly concave, differentiable benefit (felicity) functions, we develop the theory in the context of functions that are supported at the golden-rule consumption and are not necessarily concave. Through a non-interiority condition on the set of zeroes of a resulting "discrepancy function," we show the equivalence of finitely-maximal, maximal, minimal value-loss and optimal programs, and thereby answer questions left open by Brock and Mitra. Our synthesizing criterion is new to the capital theory literature, and in the concave setting, proves to be necessary and sufficient for the asymptotic convergence of good programs.

JEL classification: C62; D90; Q23

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1. Introduction

In 1986, Mitra and Wan [36] place the economics of forestry, developed by Faustmann, Wicksell, Ohlin and Samuelson, squarely within the modern theory of intertemporal allocation in discrete time, developed by Gale, McKenzie and Brock. They consider the tradition in forestry management according to which, in the words of Samuelson [49, p. 146], "the goal of good policy is to have sustained forest yield, or even maximum sustained yield somehow defined." In the framework of Ramseyian dynamics, they show that starting from any initial forest configuration, whether the optimally-managed forest converges over time to the forest with the maximum sustained yield, the "golden-rule forest" so to speak, or oscillates around it, depends on whether the benefit (objective) function is respectively strictly concave or linear.[1] Subsequent work has focused entirely on convergence in the strictly concave case with the obvious implication that there is nothing further to be said for the linear case,[2] and with the literature moving on to other concerns: to the extensions of the model to the discounted setting; to the analyses of its many variations; and to the effective exploitation of its basic methodology to other apparently unrelated issues such as the 'choice of technique' in development planning.[3]

This is a little puzzling in view of the rather strict dichotomous nature of the Mitra–Wan analysis: one set of results for linearity and another for strict concavity. This polarity of conception has pervaded virtually all of subsequent investigations.[4] The question has remained open as to what one ought to expect for a concave objective function. When the question is posed in this way, there is perhaps the expectation that a synthetic result can be obtained simply by somehow putting together the separate analyses of the two cases. At one level this intuition is incontestable, but what it misses is the precise formulation of a result for the concave setting that recovers the strictly concave and the linear cases as corollaries. What is a surprise, and perhaps the principal finding of this paper, is that in the pursuit of such a result, the assumption of concave felicities can be dispensed with altogether, and a substantial part of the theory developed in the setting of upper semicontinuous benefit (felicity) functions.[5] In such a setting, we offer a non-interiority condition that is sufficient for the asymptotic convergence of good programs to the golden-rule

[1] Throughout this essay, we shall use the words felicity function, benefit function and objective function synonymously, and use the second specifically in the context of the forestry model.

[2] This is true of Mitra's subsequent work [31,32] as well as that of the authors themselves [22].

[3] For the discounted setting, see Mitra and Wan [35] and Salo and Tahvonen [46,47]; for variants, see Mitra, Ray and Roy [37], Salo and Tahvonen [48], Cominetti and Piazza [10], Rapaport et al. [43] and Piazza [41,42]; and for development planning, see Khan and Mitra [19,20]. Commenting in their 2006 survey on the usefulness of the price-supported golden-rule in studying long run dynamic behavior of optimal programs in the undiscounted case, Mitra and Nishimura [34, Section 7] see the two papers, [36] and [20], together and note the "effective demonstration in applications of the theory to study the Faustmann solution in the forest management problem and in the analysis of the choice of technique in development planning."

[4] See [7,11,21] and their references in addition to the references in the paragraph above and in footnote 3.

[5] See Peleg [40] for an early emphasis on upper semicontinuous objective functions. We are grateful to an anonymous referee for drawing Peleg's work to our attention. As emphasized in the sequel, it is only in the analysis of finitely-maximal programs that we need to assume continuity.

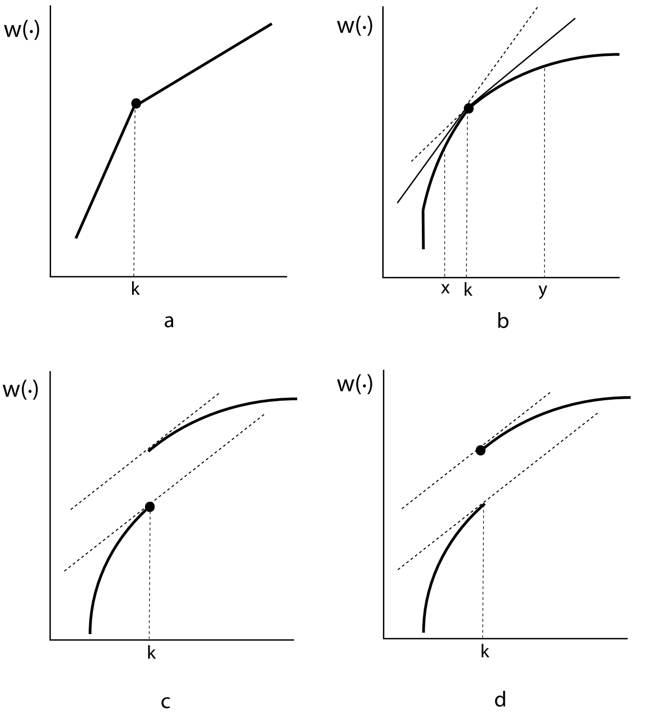

Fig. 1. Two notions of strict concavity of a function.

forest configuration, and when the benefit function is concave, though not necessarily differentiable, the condition proves to be both necessary and sufficient.

The non-interiority condition that we propose is simple and substantive enough that a basic intuitive explanation can be given to a general reader without going into technical details. To begin with, the idea of a strictly concave function is well-understood to be a global property of the function: the function w on the reals in Figs. 1a and 1b is *not* strictly concave. However, in his seminal study of intertemporal optimization, Gale [14, p. 16] formulated a notion of strict concavity of a function by privileging a particular point (the golden-rule stock) in its domain, and by defining the notion of strict concavity *at that point*. Thus, in Fig. 1b, the function w is strictly concave at k in the sense of Gale since its graph lies above any chord drawn *from* k; this condition is not fulfilled for w depicted in Fig. 1a.[6] The principal analytical innovation in this work consists in a notion of *strict concavity at a point* that goes beyond Gale's, and according to which the function depicted in Fig. 1a is also strictly concave at k. Such a notion of local concavity at a point k corresponds, in the case of a concave function, to a function whose graph lies strictly above any chord connecting two points that *represent* k, which is to say any two points whose convex combination is k. For concave functions, the notion of non-interiority presented in this paper is equivalent to the requirement of strict concavity at a point in the sense

[6] See Assumption 5 in [14], and the role it plays in his "main existence theorem," Theorem 9. Also see [15]. Gale also presented an example to show that strict concavity of a point in his sense does not imply strict concavity in the neighborhood of the point; see [14, Section 9].

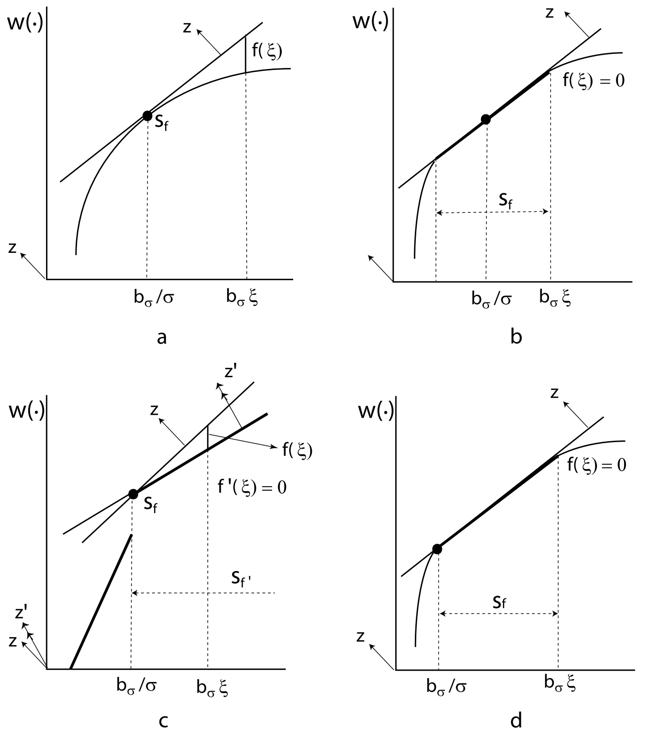

Fig. 2. The discrepancy function $f(\cdot)$.

of Roberts and Varberg [45, p. 268].[7] What is surprising, and allows a fuller appreciation of the principal result of this work, is that we can go beyond concave objective functions to upper semicontinuous functions that can be "supported" at k, as depicted in Fig. 1d (Fig. 1c is not upper-semicontinuous). The relevance of these considerations to optimization theory is obvious, and merits a further explanation.

The basis of our generalization is a natural "discrepancy function" f that consists of the difference between the felicity function and its support at a particular point k. Briefly stated, and again leaving the details to the sequel, we require that the point k not to be an interior point of the convex hull of the set S_f of its "zeros." When the function is concave, the set S_f is already convex, and when it is strictly concave in the conventional sense, the set is a singleton, and therefore the non-interiority condition is automatically fulfilled! In Fig. 2, based on a situation where a tree of age i $(i = 1, 2, \dots, n)$ yields b_i units of timber when chopped down (the biomass coefficient), $k = b_\sigma/\sigma$ is the golden-rule timber yield, $\sigma \in \{1, 2, \dots, n\}$, and $w(\cdot)$ the benefit function on yields, it is only in Fig. 2b that the condition we propose is not fulfilled. Fig. 2a depicts a strictly concave function, Fig. 2c an upper semicontinuous function that can be supported at the

[7] For a proof of this claim, see [33]. Note also that Roberts–Varberg do not necessarily assume a convex domain of the function for their definition. One may also point out to the interested reader that for concave functions, Gale's definition is stronger than that of Roberts–Varberg, and the affiliated, though different, notion in Bonsall [5]; routine verifications available from the authors on request. The authors are grateful to Tapan Mitra for enlightening correspondence on these issues.

golden-rule stock k, and Figs. 2b and 2d concave functions. Note that there is no requirement that the support be unique, as in Fig. 2c. Also note that we make this assumption on the set of timber yields rather than on the set of "today-tomorrow" forest configurations, which is to say, on the space of consumption levels (flows) rather than that of capital stocks. As such, there is some slight retreat in our analysis from the Gale–McKenzie–Brock reduced form formulation, with its exclusive focus on state variables, to a primitive assumption involving the control variables. In any case, with this condition in hand, one can establish the asymptotic convergence of good programs, a fundamental notion due to Gale [14], if this non-interiority assumption holds, and with concave benefit functions, *if and only if* it holds.[8] Thus, the corresponding results in [36] can be generalized to a considerable extent, and the basic theory rather comprehensively outlined. However, it is worth underscoring in summary that we conceive of the unified analysis presented here as much a contribution to the economics of forestry as to the general theory of intertemporal allocation of resources.

One final observation regarding the relevance of the analysis presented here to the general theory of intertemporal allocation of resources. In departing from the notational framework of [36], and on treating a forest configuration as simply a point in an $(n-1)$-dimensional simplex,[9] and the model as the pair (w, b), $b = (b_1, b_2, \dots, b_n)$, we see that the two basic assumptions of the general theory, "inaction" and "free disposal," are not fulfilled.[10] The absence of these assumptions is of course circumvented in the literature,[11] but by making it explicit, we can dispense with all assumptions on the biomass coefficients[12] other than the Brock–Mitra–Wan assumption of a unique golden-rule configuration. This is the assumption of the existence of $\sigma \in \{1, 2, \dots, n\}$ such that $(b_\sigma/\sigma) > (b_i/i)$ for all $i \neq \sigma$. In this connection, Zaslavski's recent rewriting [53] of the general theory in the context of compact metric spaces, and without any linear and ordered structures, is also of interest, and serves as a useful point of introduction to our work. Despite its apparent generality (no convexity assumptions on the technology or concavity assumptions on the benefit function), the sufficient conditions isolated by Zaslavski, and in particular his emphasis on the interiority assumption, do not translate to the forestry model. In particular, what he takes as one of the hypothesis of his results, the asymptotic convergence of good programs, is precisely what we need to prove as a consequence of our non-interiority condition. The fact that it also implies this condition is an added and satisfying bonus.

While the unification of the strictly concave and linear objective functions presented here is a primary motivation behind this essay, we are also intrigued by recent axiomatic investigations of intergenerational equity, and the espousal of, what we call here, finite-maximality as

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[8] To anticipate somewhat, from a technical point of view, the novelty of our approach consists in showing that every good program converges to the solution set of a finite-dimensional optimization problem, see Proposition 5.2 and Lemma 5.3. And we prove later on that this set collapses to the golden-rule stock when the non-interiority condition holds, see Lemmas 5.4 and 5.5 below. When restricted to the concave case, the asymptotic convergence of every good program is equivalent to the non-interiority condition, as brought out in Theorem 5.2. This emphasis on asymptotic convergence of good programs is also evident in [32,51], and of course in [53]; also see [54].

[9] This notational simplification is proposed in [46] and pursued in [18,22].

[10] In addition to [26,27], see the more recent statements of this theory in Mitra [30] and Dana et al. [12].

[11] In addition to [36] and [22], this is also implicit in treatments of the multi-specie extensions of the Mitra–Wan forest in [10], and in [41,42].

[12] As will emerge in the sequel, we do not use Assumptions A.1, A.2, A.4 and A.7 in [36] at all. Assumptions A.1, A.2 and A.7 are conditions on the biomass coefficients, while A.4 establishes the continuity and differentiability of the benefit function.

an analytical substitute for the overtaking criterion.[13] This point of view is adduced in the context of the Ramsey–Cass–Koopmans aggregate growth model by Basu and Mitra [31], and in the context of the forestry model, by Mitra [31].[14] Using the latter as a first exploratory step towards multi-sectoral capital theory, the equivalence of finitely-maximal and maximal programs is shown under concave, continuous differentiable and strictly mid-concave benefit functions, and the latter is simply a rephrasing of strict concavity.[15] The open question as to whether these hypotheses could be removed under an alternative method of proof is completely resolved.[16] Under an alternative proof, one that does not invoke full duality theory associated with maximal programs and the consequent utilization of the Arrow–Hurwicz–Uzawa constraint qualification, we show that an equivalence theorem holds under the larger class of benefit functions that we isolate here, and perhaps more importantly, the theorem dovetails to offer an even more satisfactory unified conception in which finitely-maximal, maximal, minimal value-loss and optimal programs are all identical, and with the added payoff that the existence of one implies that of all the others. As such, our primary and secondary motivations coalesce in one theorem – Theorem 7.2 below – which can perhaps be regarded as the principal contribution of this essay.

The remainder of the paper is as follows. In Section 2, we present a brief recapitulation of the general theory, and apply its basic conceptual vocabulary in Section 3 to the forestry model and its golden-rule configuration. In keeping with the emphasis given to good programs in [32] and in [53,54], Sections 4 and 5 concern good programs: the first deals with existence and characterization, and the second with the question of asymptotic convergence. Section 5 presents the non-interiority condition on the "discrepancy function" and alternative characterizations and translations of the von Neumann facet. Section 6 concerns optimal and maximal programs, and is devoted to showing the equivalence between maximal, optimal and minimal value-loss programs when the non-interiority condition holds, and to spelling out the relations between these concepts that are valid in general.[17] In this connection, it is worth underscoring that our results answer a question that Brock left open in 1970 and which Peleg resolved negatively in 1973 for the general model. Our resolution, albeit for the Mitra–Wan tree farm, is clearly opposed to the Brock–Peleg initial intuition, based on Brock's example of a "von Neumann economy" not having an optimal program,[18] and is in keeping with the recent emphasis in [51–53]. Finally, we can

[13] As referred to in footnote 17 below, terminological confusion is endemic to the subject.

[14] For this rich, and growing literature, see [2,3] and the volume [44] and its references. In particular, it is noted in [31, p. 139] that "in the context of the forestry model, one can completely dispense with the more restrictive overtaking criterion." Also see Section 4.1 and footnote 29 in [3].

[15] See Appendix A below.

[16] See [31, footnote 25] which states in the context of the continuous differentiability assumption on the objective function, "While this assumption is crucial to our method of investigation (duality theory), it is not clear whether it is indispensable for the results of the next section on the relation between maximal and optimal paths. It would seem that a "primal route" to those results should be possible."

[17] There is an important issue of rather unfortunate terminology here; optimal programs in the work of [19,20] are referred to in this work as *maximal* programs, and strongly optimal programs as simply *optimal programs*. We take our terminology from McKenzie [27, p. 256], and it is also used in [51]. It is perhaps also worth mentioning that optimal programs are referred to as *overtakingly optimal* in [53], and the terminology of maximal programs is given a different meaning in [31]. This terminological proliferation is already evident in Brock [6]. Also see footnote 13 above.

[18] Brock wrote "One might ask if it is possible to strengthen Theorem 1 [the theorem on the existence of maximal programs] by showing existence of an optimal programme. We do not know the answer to this question but we can find an example [where the answer is negative.] We doubt that it is possible to replace "weakly maximal" by "maximal" ["maximal" in Brock's terminology and not this paper's]." Subsequently, Peleg [40] removed all these doubts by observing that a "further examination of Brock's example shows that his existence theorem cannot be strengthened – a fact

characterize the set of periodic programs with zero accumulated loss, and through them the optimal Faustmann policy, when this non-interiority condition does not hold. We again underscore the fact that these results, and the analyses underlying them, are set in a context of functions that are not assumed to be concave or even continuous,[19] leave alone differentiable, provided they are supported at (b_σ/σ), the consumption at the golden-rule configuration. Section 7 considers finitely-maximal programs, and presents the unification theorem. Section 8 concludes the paper with remarks pertaining to future work that we contemplate for both the deterministic and stochastic settings. The fact that strict concavity is identical to concavity *and* strict mid-concavity is relegated to Appendix A.

2. The general theory: a recapitulation

The Gale–McKenzie reduced form model [14,25] has by now been comprehensively surveyed, and received handbook and textbook treatment; see [26,27,30,12]. For a reader wanting only a brief and basic introductory outline, Brock [6] remains the relevant reference. In the setting of a finite-dimensional Euclidean space, Brock defined maximal and optimal programs of an infinite period optimization problem, and under a uniqueness assumption of the solution of a static version of this problem, showed the existence of maximal programs.[20] Through an example, he also showed the non-existence of an optimal program. Brock's existence proof is interesting on at least two counts: (i) his reliance on a uniqueness assumption, in addition to compactness of the constraint set, and continuity on it of the objective function, (ii) his simultaneous characterization of maximal programs as necessarily implying suitably-defined minimum aggregate value-loss. The methodology of his proof, as well as his counterexample, have proved influential especially in bringing out the intertwined nature of the issues of existence and characterization. Thus, Mitra–Wan in the context of their forestry model, and Khan–Mitra in the context of the RSS model, limit themselves to maximal (as opposed to optimal) programs, and use their existence proofs to characterize Faustmann policies, on the one hand, and Stiglitz policies, on the other; see [36,20].

In this brief recapitulation of the general theory, we begin with some preliminary notation. Let $\mathbb{N}$ be the set of non-negative integers and $\mathbb{R}$ ($\mathbb{R}_+$) be the set of real (non-negative) numbers. In this section, we shall work in a compact metric space (X, ρ). Let $\Omega \subseteq X \times X$ be the (stationary) technology and $u : X \times X \to \mathbb{R}$ the (stationary) felicity or benefit function. We shall assume that Ω is non-empty and closed, and u a bounded upper semicontinuous function. This pair of objects (Ω, u) represents the model, and we present two of the concepts that are basic to the theory.

Definition 2.1. A sequence $\{x(t)\}_{t=0}^{\infty} \subset X$ is called a *program* if $(x(t), x(t+1)) \in \Omega$ for all integers $t \geqslant 0$. For any natural number T, a sequence $\{x(t)\}_{t=0}^{T} \subset X$ is called a *T-program* if $(x(t), x(t+1)) \in \Omega$ for all integers $0 \leqslant t \leqslant T-1$. For any $x \in X$, we shall say that a program $\{x(t)\}_{t=0}^{\infty}$, or a T-program $\{x(t)\}_{t=0}^{T}$, is a *program*, or a *T-program from* $x \in X$, if $x(0) = x$.

which escaped Brock's notice." Brock's Example 2.1 has dominated subsequent treatments in ensuring the exclusion of the optimality criterion.

[19] It is of course the lack of concavity that leads us to emphasize the absence of an explicit continuity assumption. In this connection, also see footnote 16 above. In the context of piecewise linearity, threshold effects arising from non-differentiable benefit (felicity) functions have recently been re-emphasized by [16].

[20] Footnotes 17 and 18 above are relevant here.

We suppose, as in the literature taking its lead from Ramsey (1928), that future welfare levels are treated like current ones in the planner's objective function.

Definition 2.2. A program $\{x^*(t)\}_{t=0}^{\infty}$, is *optimal* if for any program $\{x(t)\}_{t=0}^{\infty}$, such that $x(0) = x^*(0)$ we have

$$\lim_{T\to\infty}\sup \sum_{t=0}^{T} u\big(x(t), x(t+1)\big) - u\big(x^*(t), x^*(t+1)\big) \leqslant 0.$$

In a recent paper, Zaslavski [53] exercises Occam's razor, and presents results which do not rely on any convexity assumptions on the technology or concavity assumptions on the felicity function. In particular, they do not rely on any "free disposal" or "inaction" assumptions. We present two of these results in an attempt to bring out how far the theory can proceed only with a metric structure, though by the assumption of hypotheses that are either not fulfilled in the Mitra–Wan model or whose verification represents precisely the analysis that needs to be carried out. Despite the direct inapplicability of these theorems to our context, a point more fully established in the next section, they are useful as a parsimonious introduction to the general theory, and in highlighting three substantive points: (i) an emphasis on interiority,[21] and on the implicit retention of Brock's uniqueness assumption, (ii) an emphasis on good programs and their convergence, as opposed to their Cesàro summability, as in [6],[22] and, (iii) a return to the consideration of optimal (as opposed to maximal) programs.

For any natural number T, let

$$\sigma(u, \Omega, T) = \sup\left\{\sum_{t=0}^{T-1} u\big(x(t), x(t+1)\big)\colon\ \big\{x(t)\big\}_{t=0}^{T} \text{ is a } T\text{-program}\right\},$$

where we follow the convention that the supremum of an empty set is negative infinity.

We now reproduce the first substantive assumption on the pair (Ω, u) in [53].

Assumption 2.1. There exists $(\hat{x}, \hat{x}) \in \Omega$ and a constant $c > 0$ such that (i) $(\hat{x}, \hat{x})$ is an interior point of Ω, (ii) $u(\cdot,\cdot)$ is continuous at $(\hat{x}, \hat{x})$, (iii) $Tu(\hat{x}, \hat{x}) + c \geqslant \sigma(u, \Omega, T)$ for all natural numbers $T \geqslant 1$.

It is easy to see that under Assumption 2.1, for each natural number T and each T-program,

$$\sum_{t=0}^{T-1} u\big(x(t), x(t+1)\big) \leqslant \sigma(u, \Omega, T) \leqslant Tu(\hat{x}, \hat{x}) + c.$$

This implies that the sequence $\{g_x(T)\}_{T=1}^{\infty}$ is bounded or diverges to negative infinity, where

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[21] Zaslavski's interiority assumption is made on the technology, and as such is very different from that our assumption developed below.

[22] For Cesàro summability, the reader is referred to the Wikipedia entry on *Cesàro mean*. Following [6], this property has subsequently referred to in the literature as the "average turnpike property;" see, for example, [36] and [20] and also Lemma 6.2 and footnote 30 below. Other than in this sentence, and in keeping with the discussion in [23], we avoid the term *turnpike* or *average turnpike* in this paper.

$$g_x(T) \equiv \left\{ \sum_{t=0}^{T-1} u\big(x(t), x(t+1)\big) - T u(\hat{x}, \hat{x}) \right\}.$$

For the next assumption, we need the notion of good programs originally due to Gale [14].

Definition 2.3. A program $\{x(t)\}_{t=0}^{\infty}$ is *good* if the corresponding sequence $\{g_x(T)\}_{T=1}^{\infty}$ is bounded.

We now reproduce the second substantive assumption on the pair (Ω, u) in [53]. Note that this assumption implicitly requires Brock's uniqueness assumption on the golden-rule stock, and is of course not an assumption on the primitives on (Ω, w) of the model.[23]

Assumption 2.2. Any good program converges to $\hat{x}$ as defined in Assumption 2.1.

We can now present the basic existence result from [53, Theorem 2.2]. The invocation of the hypothesis of asymptotic convergence of good programs for a result on the existence of optimal programs should be particularly noticed.

Theorem 2.1. *Under Assumptions 2.1 and 2.2, for any z in X, if there exists a good program $\{x(t)\}_{t=0}^{\infty}$ with $x(0) = z$, there exists an optimal program $\{x^*(t)\}_{t=0}^{\infty}$ with $x^*(0) = z$.*

Next, we present a necessary and sufficient condition from [53, Theorem 2.4] for the characterization of optimal programs. But for this, we first need to consider, for any given $M > 0$, the set X_M of initial stocks $x \in X$ for which there exists a program $\{x(t)\}_{t=0}^{\infty}$, $x(0) = x$, and

$$\sum_{t=0}^{T-1} \big(u\big(x(t), x(t+1)\big) - u(\hat{x}, \hat{x})\big) \geqslant -M \quad \text{for all } T \geqslant 1.$$

These are stocks that lead to programs that are good "of the order" M.

We can now present

Theorem 2.2. *Under Assumptions 2.1 and 2.2, any program $\{x(t)\}_{t=0}^{\infty}$, with $x(0) \in \bigcup\{X_M\colon M \in (0, \infty)\}$,[24] is optimal if and only if* (i) *$\lim_{t\to\infty} \rho(x(t), \hat{x}) = 0$, and* (ii) *for each natural number T, and for any T-program with $\{y(t)\}_{t=0}^{T}$, with $y(0) = x(0)$, $y(T) = x(T)$,*

$$\sum_{t=0}^{T-1} u\big(y(t), y(t+1)\big) \leqslant \sum_{t=0}^{T-1} u\big(x(t), x(t+1)\big).$$

We now turn to the Mitra–Wan tree farm and show how these theorems, inspite of providing useful relevant benchmarks, are not directly applicable.

[23] The reader will have to wait till Section 3 for a formal definition of the golden-rule stock to see that the implicit definition of $\hat{x}$ in [53] is identical to it.

[24] The set $\bigcup\{X_M\colon M \in (0, \infty)\}$ is the set of initial conditions from where at least one good program starts.

3. The Mitra–Wan tree farm and its golden-rule configuration

From now on, we move from a compact metric space to an n-dimensional Euclidean space $\mathbb{R}^n$. We shall work in the $(n-1)$-dimensional simplex $\Delta = \{x \in \mathbb{R}^n_+ \colon \sum_{i=1}^n x_i = 1\}$. For any $x, y \in \mathbb{R}^n$ we denote the inner product by $xy = \sum_{i=1}^n x_i y_i$ and the supreme norm of x by $\|x\|_\infty$. We will consider the distance induced by the supreme norm $\operatorname{dist}(x, x') = \|x - x'\|_\infty$.

In addition to its original formulation [35,36], an outline of the Mitra–Wan forestry model is also available in [31], and of the special dual-aged case, in [30,32]. Here we depart from the original specification and work with the reformulation presented in [46] and pursued in [18,22, 47]. Under this specification, the model consists simply of the pair (b, w), where b is a non-negative vector of biomass coefficients $(b_1, \dots, b_n)$, and $w : [0, \infty) \to \mathbb{R}$ the benefit (felicity) function of timber yields. A forest (farm) configuration is an element of Δ, representing the fact that trees of ages ranging from one to n cover completely a homogeneous plot of land of normalized unit size. We invite the reader to compare this parsimonious conception with that of [36], and to note that we do not use their timber-content function $f(\cdot)$, and make no assumptions on the biomass coefficients other than the following Brock–Mitra–Wan uniqueness condition.

Standing Hypothesis (BMW). There exists $\sigma \in \{1, \dots, n\}$ such that $(b_\sigma/\sigma > b_i/i)$ for all $i \in \{1, \dots, n\} \setminus \{\sigma\}$.

In addition to this, we very much follow the original conception and assume that there are no costs of plantation, that the timber content per unit of area is related only to the age of the trees, and that n is the age after which a tree dies or losses its economic value. However, one difference should be noted. In their treatment, Mitra–Wan take N to be the age at which the biomass per unit of land is maximized, claiming that "for any reasonable objective function for the economy, trees will never be allowed to grow beyond age N; we therefore take this as a condition of feasibility itself."[25] It is this reasoning that allows the authors to limit themselves to an N-dimensional state vector. However, given the fact that a concavity benefit function favors a homogeneously configured forest, the planner may well adopt the trade-off of postponing harvesting beyond age N in order to reshape the forest into a more homogeneous state. We circumvent this by simply assuming n to be the age at which a tree dies.[26]

In summary, for each period $t \in \mathbb{N}$, we denote $x_i(t) \geqslant 0$, $i = 1, \dots, n$, the surface area occupied by trees of age i at time t. We represent the state of the forest by the vector $x(t) = (x_1(t), \dots, x_n(t)) \in \Delta$. At every stage the planner must decide how much land to harvest of every age-class, $c(t) = (c_1(t), \dots, c_n(t))$ where $c_i(t) \in [0, x_i(t)]$. As we know that after the age n, a tree has no value, we assume that $c_n(t) = x_n(t)$ for all t. By the end of period $t+1$, the state will be exactly

$$x(t+1) = \left(\sum_{i=1}^n c_i(t), x_1(t) - c_1(t), \dots, x_{n-1}(t) - c_{n-1}(t) \right).$$

This leads us to rewrite Definition 2.1.

[25] See [36, p. 232]. The same point is made in [31, Section 4, Paragraph 5].

[26] This is simply a somewhat subtle point of interpretation; the technicalities of the two analyses remain the same.

Definition 3.1. A sequence $\{x(t)\}_{t=0}^{\infty}$ is called a program if for each $t \geqslant 0$,

$$\begin{cases} x(t) \in \Delta, \\ x_{i+1}(t+1) \leqslant x_i(t), \quad i = 1, \dots, n-1. \end{cases} \tag{1}$$

We can now define the transition possibility set Ω as the collection of pairs $(x, x') \in \Delta \times \Delta$ such that it is possible to go from the state x in the current period (today) to the state of the forest x' in the next period (tomorrow) fulfilling relations (1). Namely,

$$\Omega = \{(x, x') \in \Delta \times \Delta\colon\ x_i \geqslant x'_{i+1} \quad \text{for all } i = 1, \dots, n-1\}.$$

Note, in passing, that the transition set Ω is convex, closed and stationary, and as such complications arising from non-convexity and non-stationarity can only originate from the benefit function.

Definition 3.2. The vector of harvests to perform this transition is given by the function $\lambda : \Omega \to \mathbb{R}^n_+$,

$$\lambda(x, x') = (x_1 - x'_2, x_2 - x'_3, \dots, x_{n-1} - x'_n, x_n).$$

In addition, it is easy to see that $(x, x') \in \Omega \Leftrightarrow (x, x') \in \Delta \times \Delta$ and $\lambda(x, x') \geqslant 0$.

The preferences of the planner are represented by a benefit (felicity) function, $w : [0, \infty) \to \mathbb{R}$ which is assumed to be non-decreasing and upper semicontinuous. Define for any $(x, x') \in \Omega$ the function $u(x, x')$ as

$$u(x, x') = w(bc) \quad \text{where } c = \lambda(x, x'). \tag{2}$$

We also assume that the function w is strictly increasing at (b_σ/σ) and supported at the point (b_σ/σ), which is to say that there is $z > 0$ satisfying:

$$w(y) \leqslant w\left(\frac{b_\sigma}{\sigma}\right) + z\left(y - \frac{b_\sigma}{\sigma}\right) \quad \text{for all } y \in \mathbb{R}_+. \tag{3}$$

Remark 3.1. This assumption on w is weaker than the classical strict concavity. Indeed, it is well known that the upper sub-differential of a concave function, $\partial^+ w(c)$, is non-empty for every c in the interior of its domain. Furthermore, for every $c > 0$, $\partial^+ w(c)$ is either the singleton $\{w'(c)\}$, whenever w is differentiable at c, or the closed bounded interval whose extremes are the side derivatives of w: $[w'_+(c), w'_-(c)]$. Moreover, the fact that w is non-decreasing and strictly concave yields $\partial^+ w(c) \subset \mathbb{R}_{++}$ for all $c > 0$. Finally, any $z \in \partial^+ w(b_\sigma/\sigma)$ fulfills (3) and we will see in the sequel that the selection of z is indifferent to our conclusions.

We now turn to the specification of the golden-rule forest configuration, and this is a good point to relate the analysis presented below to the antecedent literature. Note that the specification of the technology Ω precludes Brock's "inaction" and "free disposal" assumptions, as well as Zaslavski's interiority condition. The set Ω has no interior point in $\mathbb{R}^{2n}$ and the natural coordinate pre-order in this space does not apply to it.[27] Furthermore, Mitra–Wan appeal to

[27] To endow Δ with a pre-order, the natural way to proceed would be to consider a *reduced,* $(n-1)$*-dimensional state* of the forest, like for example: $z = (x_2, \dots, x_n)$, using the area constraint to deduce the area occupied by trees of age 1,

the differentiability of the benefit function to provide directly the golden-rule forest configuration and the golden-rule prices associated with it.[28] We provide a self-contained argument to characterize the golden-rule forest configuration in Theorem 3.1.

Definition 3.3. A *golden-rule stock* $\hat{x} \in \mathbb{R}^n_+$ is such that $(\hat{x}, \hat{x})$ is a solution to the problem:

$$maximize_{(x,x)\in\Omega} u(x,x) = maximize_{(x,x)\in\Omega} w\big(b\lambda(x,x)\big).$$

Definition 3.3 coincides with the definition provided by Mitra and Wan [36]. Set $\hat{p} \in \mathbb{R}^n_+$, $\hat{p} = z\frac{b_\sigma}{\sigma}(1, 2, \dots, n)$.

Next, we define the function[29] $\delta : \Omega \to \mathbb{R}$.

Definition 3.4. The *value-loss* associated with any $(x, x') \in \Omega$ is given by

$$\delta(x, x') = w\left(\frac{b_\sigma}{\sigma}\right) - w\big(b\lambda(x, x')\big) - \hat{p}(x' - x).$$

It is easy to see that the function $\delta(\cdot,\cdot)$ is lower semicontinuous and the following lemma proves that $\delta(x, x') \geqslant 0$ for any $(x, x') \in \Omega$, and also determines a disaggregated lower bound of the value-loss function that will be used afterwards in the characterization of the von Neumann facet. The non-negativity of the value-loss function is already established in [36, Lemma 3.1] when w is differentiable and concave.

Lemma 3.1. *For any $(x, x') \in \Omega$ we have*

$$\delta(x, x') \geqslant z\left[\sum_{i=1}^{n-1}\left(\frac{b_\sigma}{\sigma} - \frac{b_i}{i}\right)i\big(x_i - x'_{i+1}\big) + \left(\frac{b_\sigma}{\sigma} - \frac{b_n}{n}\right)nx_n\right] \geqslant 0. \tag{4}$$

Proof. Due to (3), we know that

$$w(y) \leqslant w\left(\frac{b_\sigma}{\sigma}\right) + z\left(y - \frac{b_\sigma}{\sigma}\right) \quad \text{for all } y \in \mathbb{R}.$$

In particular, taking $y = b\lambda(x, x')$ we get

$$\delta(x, x') \geqslant w\left(\frac{b_\sigma}{\sigma}\right) - w\left(\frac{b_\sigma}{\sigma}\right) - z\left[\sum_{i=1}^{n-1} b_i\big(x_i - x'_{i+1}\big) + b_n x_n - \frac{b_\sigma}{\sigma}\right] \\ - \sum_{i=1}^{n} z\frac{b_\sigma}{\sigma} i\big(x'_i - x_i\big)$$

$x_1 = 1 - \sum_{i=2}^n x_i$. We can now work with the pre-order defined by $z \geqslant z'$ if $z_i \geqslant z'_i$ for all $i = 2, \dots, n$. To see that the new formulation of the model does not fulfill the free disposal assumption, consider the state $z = (0, \dots, 0)$ representing the case where all the trees are of age 1. It is possible to go from the state z today to the state $y = (1, \dots, 0)$ tomorrow (all the trees are of age 2). Any state $z' \neq z$, will satisfy $z' \geqslant z$, but the transition from z' to y will be not possible.

[28] In addition to Assumption 2.1 above, see [6, Assumption 1] and [36, Lemma 3.1]. Note that Brock's "sufficiency" assumption [6, Assumption 2] is automatically fulfilled in our context.

[29] A warning to the reader that the function $\delta(\cdot,\cdot)$ is to be distinguished from the real number δ, typically assumed to be positive.

$$= z\left[-\sum_{i=1}^{n-1} b_i\left(x_i - x'_{i+1}\right) - b_n x_n + \frac{b_\sigma}{\sigma} - \frac{b_\sigma}{\sigma} \underbrace{\sum_{i=1}^{n} i\left(x'_i - x_i\right)}_{(A)}\right].$$

Rearranging the last summation we get:

$$\begin{aligned}
(A) &= \sum_{i=1}^{n} i x'_i - \sum_{i=1}^{n} i x_i = \sum_{i=0}^{n-1} (i+1) x'_{i+1} - \sum_{i=1}^{n} i x_i \\
&= \sum_{i=0}^{n-1} x'_{i+1} + \sum_{i=1}^{n-1} i\left(x'_{i+1} - x_i\right) - n x_n \\
&= 1 - n x_n + \sum_{i=1}^{n-1} i\left(x'_{i+1} - x_i\right).
\end{aligned} \tag{5}$$

We substitute this expression in the previous inequality

$$\begin{aligned}
\delta\left(x, x'\right) &\geqslant z\left\{-\sum_{i=1}^{n-1} b_i\left(x_i - x'_{i+1}\right) - b_n x_n + \frac{b_\sigma}{\sigma} - \frac{b_\sigma}{\sigma}\left[1 - n x_n + \sum_{i=1}^{n-1} i\left(x'_{i+1} - x_i\right)\right]\right\} \\
&= z\left[\sum_{i=1}^{n-1}\left(\frac{b_\sigma}{\sigma} - \frac{b_i}{i}\right) i\left(x_i - x'_{i+1}\right) + \left(\frac{b_\sigma}{\sigma} - \frac{b_n}{n}\right) n x_n\right] \geqslant 0,
\end{aligned}$$

where the last inequality follows easily by observing that $(\frac{b_\sigma}{\sigma} - \frac{b_i}{i}) \geqslant 0$ and $(x, x') \in \Omega$ and $z > 0$. □

The following theorem is basic to the subject. A proof is provided by [36, Theorem 3.1] using slightly stronger hypothesis on the biomass coefficients. We provide a proof suitable to our framework, one that does without the already mentioned differentiability assumptions on the benefit function w.

Theorem 3.1. *There exists a unique golden-rule stock*

$$\hat{x} = \Big(\underbrace{\frac{1}{\sigma}, \dots, \frac{1}{\sigma}}_{\sigma}, 0, \dots, 0\Big).$$

Proof. First observe that $\hat{c} = \lambda(\hat{x}, \hat{x})$ is such that $\hat{c}_\sigma = \frac{1}{\sigma}$ and $\hat{c}_i = 0$ for all $i \neq \sigma$, hence $b\hat{c} = \frac{b_\sigma}{\sigma}$. The proposition above yields

$$\delta(x, x) = w\left(\frac{b_\sigma}{\sigma}\right) - w\left(b\lambda(x, x)\right) - \hat{p}(x - x) \geqslant 0 \quad \text{for all } (x, x) \in \Omega$$

implying immediately that $u(\hat{x}, \hat{x}) = w(\frac{b_\sigma}{\sigma}) \geqslant w(b\lambda(x, x))$ for all $(x, x) \in \Omega$.

It is left to see that $\hat{x}$ is the unique golden-rule stock. Let us suppose that there is $x \neq \hat{x}$ solution to the problem stated in Definition 3.3. Using that w is strictly increasing at $\frac{b_\sigma}{\sigma}$ we get

$$\frac{b_\sigma}{\sigma} = b\lambda(x, x) = \sum_{i=1}^{n-1} b_i(x_i - x_{i+1}) + b_n x_n. \tag{6}$$

On the other hand, condition $(x,x) \in \Omega$ forces $(x_i - x_{i+1}) \geqslant 0$, which together with (BMW) yields

$$b\lambda(x,x) = \sum_{i=1}^{n-1} \frac{b_i}{i} i(x_i - x_{i+1}) + \frac{b_n}{n} n x_n \leqslant \sum_{i=1}^{n-1} \frac{b_\sigma}{\sigma} i(x_i - x_{i+1}) + \frac{b_\sigma}{\sigma} n x_n$$

with strict inequality unless $(x_i - x_{i+1}) = 0$ for all $i \neq \sigma$ and $x_n = 0$. Setting $x = x'$ in Eq. (5) we can rewrite the right hand side above to get

$$b\lambda(x,x) \leqslant \frac{b_\sigma}{\sigma}\left[\sum_{i=1}^{n-1} i(x_i - x_{i+1}) + n x_n\right] = \frac{b_\sigma}{\sigma}[1 - n x_n + n x_n] = \frac{b_\sigma}{\sigma}.$$

It is easy to see that to have (6), we need $x_i = x_{i+1}$ for all $i \neq \sigma$, $x_n = 0$ and $x_\sigma - x_{\sigma+1} = \frac{1}{\sigma}$, namely, $\lambda(x,x) = \hat{c}$. From this and $(x,x) \in \Omega$ it follows that $x = \hat{x}$. □

4. On good programs: existence and characterization

As emphasized in the introduction, the principal thrust of the analysis presented in this paper revolves around the identification of a necessary and sufficient condition for the asymptotic convergence of good programs. Towards this end, we develop in this section some preliminary results concerning the existence and characterization of good programs.

For the remainder of the paper, we shall abbreviate $\{x(t)\}_{t=0}^{\infty}$ by $\{x(t)\}$ and rewrite Definition 2.3 in terms of the parameters of the forestry model.

Definition 4.1. A program $\{x(t)\}$ is called *good* if there exists $M \in \mathbb{R}$ such that for all $T \geqslant 0$, $\sum_{t=0}^{T}[w(bc(t)) - w(\frac{b_\sigma}{\sigma})] \geqslant M$, where $c(t) = \lambda(x(t), x(t+1))$. A program is *bad* if $\lim_{T\to\infty} \sum_{t=0}^{T}[w(bc(t)) - w(\frac{b_\sigma}{\sigma})] = -\infty$.

As shown in [36, Lemma 4.3], the following general result of Gale applies to the Mitra–Wan forestry model. A proof for our notational framework is available as the proof of [22, Proposition 2.1].

Proposition 4.1. *The space of programs is partitioned into good and bad programs.*

The proposition below shows an equivalent characterization of good and bad programs and its proof is available as the proof of [22, Proposition 2.2] with the usual care in regarding z as an element of the subdifferential.

Proposition 4.2. *A program $\{x(t)\}$ is good if and only if $\sum_{t=0}^{\infty} \delta(x(t), x(t+1))$ is finite, and bad if and only if $\sum_{t=0}^{\infty} \delta(x(t), x(t+1))$ is infinite.*

Next we record the existence of at least one good program from any initial state as constructed in [22, Remark 2.2].

Lemma 4.1. *There is a good program from every $x_0 \in \Delta$.*

Let $x_0 \in \Delta$. Set $\mu(x_0) = \inf\{\sum_{t=0}^{\infty} \delta(x(t), x(t+1))$: $\{x(t)\}$ is a program from $x_0\}$.

The lemma above implies that $\mu(x_0) < \infty$. We can now establish the existence of a program that attains minimum aggregate value-loss. This is a benchmark result in the literature, but the proof we present adapts an argument in [12, Proposition 1.4.2] and circumvents Cantor's diagonalization argument in [6], and following him, in [36] and [20].

Proposition 4.3. *From any $x_0 \in \Delta$ there exists a program $\{x(t)\}$ such that*

$$\sum_{t=0}^{\infty} \delta\big(x(t), x(t+1)\big) = \mu(x_0). \tag{7}$$

Proof. Let us define the functions

$$\bar{\delta} : \Delta \times \Delta \to \overline{\mathbb{R}}_+, \qquad \bar{\delta}\big(x, x'\big) = \begin{cases} \delta(x, x') & \text{if } (x, x') \in \Omega, \\ \infty & \text{else,} \end{cases}$$

and

$$\gamma : \Pi \to \overline{\mathbb{R}}_+, \qquad \gamma\big(\{x(t)\}\big) = \sum_{t \in \mathbb{N}} \bar{\delta}\big(x(t), x(t+1)\big),$$

where $\Pi = \prod_{t=0}^{\infty} \Delta$. Let $\gamma_T(\{x(t)\}_{t\in\mathbb{N}}) = \sum_{t=0}^{T-1} \delta(x(t), x(t+1))$. The function $\bar{\delta}$ is lower semicontinuous and so is γ_T with respect to the product topology, for every T. In addition $\gamma_T \leqslant \gamma_{T+1}$, hence γ is the increasing limit of l.s.c. functions, it is therefore lower semicontinuous. We know that Π is compact in the product topology and that there is at least one good program, so the domain of γ is non-empty. Then, there is a minimizer $\{x^*(t)\}$ such that $\gamma(\{x^*(t)\}) = \mu(x_0) < \infty$. □

5. On good programs: asymptotic properties

It is now well understood from the general theory that with strictly concave felicity functions, the von Neumann facet comprises only the pairs of states whose harvest corresponds to the harvest associated to the golden-rule forest configuration, and as a result, any good program asymptotically converges to the golden-rule stock. This result is established for the forestry model in [36, Lemma 6.4]. We prove the convergence of good programs towards the golden-rule stock, $\hat{x}$, not only when w is strictly concave but for a broader family of benefit functions, and towards that end, present a condition concerning the support function at the point (b_σ/σ) with slope z. Let the discrepancy function, f, be the difference between the affine function supporting w at (b_σ/σ), defined by (3), and the function w itself,

$$f(\xi) = w\left(\frac{b_\sigma}{\sigma}\right) - w(b_\sigma \xi) + z\left(b_\sigma \xi - \frac{b_\sigma}{\sigma}\right), \tag{8}$$

where, in order to simplify the subsequent presentation we have performed the change of variable $y = b_\sigma \xi$. Thanks to (3), we know that $f(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}_+$. Let $S_f \subseteq \mathbb{R}_+$ be the set where f attains zero, its global minimum. In the general case, all we know about S_f is that it is a closed set, thanks to the upper semicontinuity of w, and that $(1/\sigma) \in S_f$. In some particular cases, more can easily be said about S_f. For example:

- If w is linear, S_f is the entire $\mathbb{R}_+$ given that the only affine support of w is the function itself.

- If w is concave, S_f is a closed interval. Indeed, it is impossible to have a disconnected S_f when w is concave, we invite the reader to make a sketch to convince herself.
- If w is strictly concave, the set S_f is $\{1/\sigma\}$. In fact, the identity $S_f = \{1/\sigma\}$ corresponds to the family of strictly supported benefit functions,

$$S_f = \{1/\sigma\} \quad \Leftrightarrow \quad w(y) > w\left(\frac{b_\sigma}{\sigma}\right) + z\left(y - \frac{b_\sigma}{\sigma}\right) \quad \text{for all } y \neq \frac{b_\sigma}{\sigma}.$$

Let $S_c = \{c \in \mathbb{R}^n_+ \colon c_i = 0 \text{ for all } i \neq \sigma \text{ and } c_\sigma \in S_f\}$, the set of harvests of σ-aged trees that belong to the zeroes of the discrepancy function, and which allows us to give an alternative characterization of the von Neumann facet. This characterization will play a crucial analytical role in the sequel.

Proposition 5.1. *The von Neumann facet is*

$$\{(x, x') \in \Omega \colon \delta(x, x') = 0\} = \{(x, x') \in \Omega \colon \lambda(x, x') \in S_c\}.$$

Proof. Recall Lemma 3.1, and in particular (4) proved there:

$$\delta(x, x') \geqslant z\left[\sum_{i=1}^{n-1}\left(\frac{b_\sigma}{\sigma} - \frac{b_i}{i}\right) i\,(x_i - x'_{i+1}) + \left(\frac{b_\sigma}{\sigma} - \frac{b_n}{n}\right) n x_n\right] \geqslant 0.$$

Observing the sign of the coefficients $(\frac{b_\sigma}{\sigma} - \frac{b_i}{i})$ it is easy to conclude that

$$\delta(x, x') = 0 \quad \text{implies } x_i = x'_{i+1} \text{ for all } i \neq \sigma,\ i < n \text{ and } x_n = 0. \tag{9}$$

It remains to prove that $c_\sigma \in S_f$. Using (5) and (9) we can find a much simpler expression for

$$\sum_{i=1}^{n} i\,(x'_i - x_i) = 1 - n x_n + \sum_{i=0}^{n-1} i\,(x'_{i+1} - x_i) = 1 - \sigma\,(x_\sigma - x'_{\sigma+1}),$$

and so,

$$\begin{aligned} \delta(x, x') = 0 \quad &\Rightarrow \quad \delta(x, x') = w\left(\frac{b_\sigma}{\sigma}\right) - w(b_\sigma c_\sigma) - z b_\sigma\left[\frac{1}{\sigma} - (x_\sigma - x'_{\sigma+1})\right] = 0 \\ &\Rightarrow \quad f(c_\sigma) = 0 \quad (\text{using } (8)). \end{aligned}$$

So far we have proved $\{(x, x') \in \Omega \colon \delta(x, x') = 0\} \subseteq \{(x, x') \in \Omega \colon \lambda(x, x') \in S_c\}$. The inclusion on the other sense follows easily using (5) and (8). We leave the details to the reader. □

The following set V is essential for the rest of the exposition:

$$V = \{x \in \Delta \colon x_i \in S_f \text{ for all } i \leqslant \sigma \text{ and } x_i = 0 \text{ for all } i > \sigma\}. \tag{10}$$

Evidently, $\hat{x} \in V$. Proposition 5.1 implies that V is the set of initial conditions from where zero value-loss programs originate. Such programs must be σ-periodic. We claim that this behavior is typical in the sense that a good program converges to V. Hence, it will be of use to know when

Condition 5.1 *(Standing condition).* $V = \{\hat{x}\}$.

We will show that the condition above is necessary and sufficient to assure the asymptotic convergence of every good program to the golden-rule stock, and that the following, easier-to-check, condition is sufficient to assure Condition 5.1.

Condition 5.2 *(Non-interiority).* $(1/\sigma) \notin \operatorname{int}\operatorname{co}(S_f)$.

We have already seen that when the benefit (felicity) function is concave, S_f is a closed interval and hence the condition above can be simplified to

Condition 5.3 *(Non-interiority, concave case).* $(1/\sigma) \notin \operatorname{int} S_f$.

Of course, this last condition is assured when w is strictly concave. We will see in the sequel that Condition 5.3 is not only sufficient but also necessary in the concave case.

Before getting into the proofs of these assertions, we present an alternative characterization of V as the solution of the following optimization problem.

Proposition 5.2. *Consider the optimization problem,*

$$(P) \quad \begin{cases} \alpha = \min \sum_{i=1}^{\sigma} w(b_\sigma/\sigma) - w(b_\sigma x_i), \\ \quad s.t.\ x \in \Delta. \end{cases} \tag{11}$$

Then, $\alpha = 0$ and the solution to P, $S(P)$, is given by the set V.

Proof. We can easily see that P has a non-empty solution because its objective function is lower semicontinuous and the feasible set is non-empty and compact. Consider $x \in S(P)$. Only the first σ coordinates are involved in the minimization and w is non-decreasing, hence every $x \in S(P)$ must be of the form $x = (x_1, x_2, \ldots, x_\sigma, 0, \ldots, 0)$.

Take such x and consider the σ-periodic program that is obtained when harvesting only and completely the σ-age class at each step. The value-loss accumulated during the first σ steps is of course non-negative and can be expressed as

$$\begin{aligned} 0 \leqslant \sum_{t=0}^{\sigma-1} \delta\big(x(t), x(t+1)\big) &= \sum_{t=0}^{\sigma-1} \left[w\left(\frac{b_\sigma}{\sigma}\right) - w\big(b_\sigma x_\sigma(t)\big) - \hat{p}\big(x(t+1) - x(t)\big) \right] \\ &= \sum_{t=0}^{\sigma-1} \left[w\left(\frac{b_\sigma}{\sigma}\right) - w(b_\sigma x_{\sigma-t}) \right] - \hat{p}\big(x(\sigma) - x(0)\big) \\ &= \sum_{i=1}^{\sigma} \left[w\left(\frac{b_\sigma}{\sigma}\right) - w(b_\sigma x_i) \right] \end{aligned}$$

with equality when $x = \hat{x}$. From this, we know that $\alpha = 0$ and $\hat{x} \in S(P)$.

From the above, we also know that given any $x \in S(P)$, we have that the aggregate value-loss along σ steps of a periodic program is zero. Proposition 5.1 implies that $x_i \in S_f$ for all $i = 1, \ldots, \sigma$. Hence $x \in V$, and we have proved that $S(P) \subseteq V$.

To prove the converse consider any state $x \in V$. Consider the σ-periodic program that is obtained when harvesting only and completely the σ-age class at each step. Evidently $\lambda(x(t), x(t+1)) \in S_c$, which implies $\delta(x(t), x(t+1)) = 0$ for all t. Therefore

$$0 = \sum_{t=0}^{\sigma-1} \delta\big(x(t), x(t+1)\big)$$
$$= \sum_{t=0}^{\sigma-1} \left[w\left(\frac{b_\sigma}{\sigma}\right) - w(b_\sigma x_{\sigma-t}) \right] + \hat{p}\,\underbrace{\big(x(\sigma) - x(0)\big)}_{=0} \quad \Rightarrow \quad x \in S(P). \qquad \square$$

Remark 5.1. It is possible to have zero value-loss during σ steps even if the initial state does not belong to V. In fact, requiring $\sum_{i=1}^{\sigma} w(\frac{b_\sigma}{\sigma}) - w(b_\sigma x_i) = 0$ is stronger than the condition $\sum_{t=0}^{\sigma-1} \delta(x(t), x(t+1)) = 0$.

We illustrate this with a toy example. Consider a forest where $n = 4$, $\sigma = 2$ and $S_f = [\frac{1}{2} - \phi, \frac{1}{2}]$ for some $\phi > 0$. Consider the first three stages of the following program:

$$x(0) = \left(\frac{1}{2} - \phi, \frac{1}{2} + \phi, 0, 0\right),$$
$$x(1) = \left(\frac{1}{2}, \frac{1}{2} - \phi, \phi, 0\right),$$
$$x(2) = \left(\frac{1}{2} - \phi, \frac{1}{2}, 0, \phi\right).$$

Observe that $x(0) \notin V$ and that on the first two stages the value-loss is zero. Furthermore, observe that no matter what harvest is imposed on the third stage, there will be a positive value-loss due to the fact that $x_4(2) > 0$. This suggest a similar but stronger condition that implies $x \in V$.

Lemma 5.1. *If* $\sum_{t=0}^{n-1} \delta(x(t), x(t+1)) = 0 \Rightarrow x(t) \in V$ *for all* $t = 0, \ldots, \sigma$.

Proof. $\delta(x(t), x(t+1)) = 0$ implies that $\lambda(x(t), x(t+1)) \in S_c$ which in particular means

$$c_i(t) = 0 \quad \text{for all } i > \sigma \quad \text{and} \quad x_n(t) = 0. \tag{12}$$

We claim that this implies that $x_i(t) = 0$, $i > \sigma$, $t \leqslant \sigma$. Indeed, if there is $x_i(t) > 0$ with $i > \sigma$ and $t \leqslant \sigma \Rightarrow x_{i+1}(t+1) > 0 \Rightarrow x_n(n+t-i) > 0$ which contradicts (12).

We know then that $x(\sigma) = (x_1(\sigma), \ldots, x_\sigma(\sigma), 0, \ldots, 0)$ where $x_i(\sigma) = \sum_{j=1}^{n} c_j(\sigma - i) = c_\sigma(\sigma - i) \in S_f$ which gives $x(\sigma) \in V$.

Finally, it is easy to see that $x(t+1) \in V$ and $\delta(x(t), x(t+1)) = 0$ implies $x(t) \in V$. Then the lemma follows by backwards induction. $\square$

Consider the lower semicontinuous function $\gamma_n(\{x(t)\}_{t=0}^{n}) = \sum_{t=0}^{n-1} \bar{\delta}(x(t), x(t+1))$, defined in Proposition 4.3. By the lemma above, we know $\gamma_n(\{x(t)\}_{t=0}^{n}) = 0 \Rightarrow x(t) \in V$, $t = 0, \ldots, \sigma$. Furthermore,

Lemma 5.2. *Given* $\epsilon > 0$, *there is* $\delta > 0$ *such that if* $\mathrm{dist}(x, V) \geqslant \epsilon$ *then we have* $\gamma_n(x, x_1, \ldots, x_n) \geqslant \delta$ *for all* $(x_1, \ldots, x_n) \in \Delta^n$.

Proof. Suppose, contrary to our claim, that for every k there is $x^k(0)$ such that $\mathrm{dist}(x^k(0), V) \geqslant \epsilon$ and $\gamma_n(x^k(0), x^k(1), \ldots, x^k(n)) \leqslant \frac{1}{k}$ for at least one $(n-1)$-tuple $(x^k(1), \ldots, x^k(n))$. Of course, $(x^k(i), x^k(i+1))$ belongs to the compact set Ω, otherwise $\gamma_n(x^k(0), x^k(1), \ldots, x^k(n)) = \infty$.

We know that there must be at least one converging subsequence: $(x^{k_j}(0), x^{k_j}(1), \dots, x^{k_j}(n))$. Let $(\bar{x}(0), \bar{x}(1), \dots, \bar{x}(n))$ be its limit, by the lower semicontinuity of γ_n we have that

$$\gamma_n\big(\bar{x}(0), \bar{x}(1), \dots, \bar{x}(n)\big) \leqslant \liminf_j \gamma_n\big(x^{k_j}(0), x^{k_j}(1), \dots, x^{k_j}(n)\big) = 0$$

and the proposition above implies that $\bar{x}(0) \in V$.

On the other hand, $\mathrm{dist}(x^k(0), V) \geqslant \epsilon$ for all n implies $\mathrm{dist}(\bar{x}(0), V) \geqslant \epsilon$ and a contradiction arises proving the lemma. □

We are now in position of proving the important convergence result previously announced,

Lemma 5.3. *Every good program* $\{x(t)\}$ *is such that* $\mathrm{dist}(x(t), V) \to 0$.

Proof. It is evident that if $\{x(t)\}$ is a good program, then $\gamma_n(\{x(T+t)\}_{t=0}^n) \to 0$ when $T \to \infty$. This convergence together with the lemma above implies that $\mathrm{dist}(x(T), V) \to 0$. □

Corollary 5.1. *If* $\{x(t)\}$ *is a good program, then* $\mathrm{dist}(c(t), S_c) \to 0$.

Finally, we are led to the principal result of this section: the following strengthening of [36, Lemma 6.4] to our non-concave, non-differentiable context.

Theorem 5.1. $\{\hat{x}\} = V$ *iff any good program satisfies* $\lim_t x(t) = \hat{x}$.

Proof. The first implication follows directly from the lemma above. To prove the converse, it suffices to observe that if there is $x \neq \hat{x}$ such that $x \in V$, then there is a zero value-loss σ-periodic program originating from x. Evidently, this program is good and does not converge to $\hat{x}$. □

Given the last theorem it is of interest to know when Condition 5.1: $V = \{\hat{x}\}$ holds. We look for conditions on the primitives of the model assuring that $V = \{\hat{x}\}$. Evidently, $S_f = \{1/\sigma\}$ is sufficient to assure it. Even more, due to the area balance, conditions

$$1/\sigma = \min\{\xi\colon \xi \in S_f\} \quad \text{or} \quad 1/\sigma = \max\{\xi\colon \xi \in S_f\}$$

are also sufficient. All these can be expressed as the non-interiority Condition 5.2

$$\frac{1}{\sigma} \notin \operatorname{int} \operatorname{co}(S_f).$$

Lemma 5.4. *If the non-interiority Condition 5.2 holds then* $\{\hat{x}\} = V$.

Remark 5.2. The condition above is not necessary as the following toy example shows. Consider a forest where $n = 3$, $\sigma = 2$ and $S_f = \{\frac{1}{3}, \frac{1}{2}, \frac{3}{4}\}$. Of course $\frac{1}{\sigma} \in (\frac{1}{3}, \frac{3}{4}) = \operatorname{int} \operatorname{co}(S_f)$, so the condition above is not fulfilled. But the set V is

$$V = \{x \in \Delta\colon x_1, x_2 \in S_f \text{ and } x_3 = 0\} = \big\{(x_1, x_2, 0)\colon x_1 + x_2 = 1,\ x_1, x_2 \in S_f\big\} = \{\hat{x}\}.$$

If the benefit function is concave, then the non-interiority Condition 5.2 turns into the simpler Condition 5.3

$$\frac{1}{\sigma} \notin \operatorname{int}(S_f).$$

The following lemma brings out the fact that this simplified condition turns out to be necessary and sufficient for the satisfaction of Condition 5.1.

Lemma 5.5. *Let the benefit function w be concave. Then, the non-interiority Condition 5.3 holds if and only if $\{\hat{x}\} = V$.*

Proof. The sufficiency of the non-interiority Condition 5.3 follows directly from Lemma 5.4.

To see the necessity, suppose that it does not hold. Then there is $\phi > 0$ such that $[\frac{1}{\sigma} - \phi, \frac{1}{\sigma} + \phi] \subset S_f$, and

$$x = \left(\frac{1}{\sigma} - \phi, \frac{1}{\sigma} + \phi, \frac{1}{\sigma}, \dots, \frac{1}{\sigma}, 0, \dots, 0\right) \in V. \qquad \square$$

Theorem 5.2. *Let w be concave. Then the non-interiority Condition 5.3 holds if and only if any good program $\{x(t)\}$ satisfies $\lim_{t\to\infty} x(t) = \hat{x}$.*

Proof. If non-interiority Condition 5.3 holds, the lemma above yields $\{\hat{x}\} = V$ and then Lemma 5.3 implies that any good program satisfies $x(t) \to \hat{x}$. If it does not hold, then there exists $x \in V$, $x \neq \hat{x}$, and the σ-periodic program from x is a non-converging zero value-loss program. $\square$

Remark 5.3. If w is concave and non-differentiable at b_σ/σ then the non-interiority Condition 5.3 is assured. Indeed, as stated in Remark 3.1, the non-differentiability of w implies that $\partial^+ w(b_\sigma/\sigma)$ is not a singleton but the interval with non-empty interior: $[w'_+(c), w'_-(c)]$. If we take $z \in (w'_+(c), w'_-(c))$, it is easy to see that the associated $S_f = \{b_\sigma/\sigma\}$. If we take $z = w'_+(c)$ or $z = w'_-(c)$, then b_σ/σ has to be necessarily one of the extremes of the associated S_f.

6. On maximal and optimal programs

Next, we turn to the existence results for optimal programs, and also present an equivalence theorem that shows the equivalence of optimal, maximal and minimal value-loss programs. We distinguish carefully how far one can proceed without Condition 5.1, and indicate, as in the sections above, versions of our results that are already available in the literature.

First, we need a formal definition of maximal programs. In Section 2, the notion of an optimal program is defined for the general model, and on using Eq. (2), Definition 2.2 can be routinely transferred to the forestry model. We can then present the two basic concepts considered in this section.

Definition 6.1. A program $\{x^*(t)\}$ is *optimal* if for any program $\{x(t)\}$ such that $x(0) = x^*(0)$ we have

$$\limsup_{T\to\infty} \sum_{t=0}^{T} w\big(bc(t)\big) - w\big(bc^*(t)\big) \leqslant 0.$$

Definition 6.2. A program $\{x^*(t)\}$ is *maximal* if for any program $\{x(t)\}$ such that $x(0) = x^*(0)$ we have

$$\lim_{T\to\infty}\inf \sum_{t=0}^{T} w\bigl(bc(t)\bigr) - w\bigl(bc^*(t)\bigr) \leqslant 0.$$

Let us first see an easy technical lemma.

Lemma 6.1. *Every maximal program is good.*

Proof. Let $\{x^*(t)\}$ be a maximal program from x_0 and $\{x(t)\}$ *any* good program from x_0, i.e., there is $M \in \mathbb{R}$ such that for all $T \geqslant 0$, $\lim_T \sum_{t=0}^{T-1} w(bc(t)) - w(\frac{b_\sigma}{\sigma}) \geqslant M$. Hence, we get

$$\sum_{t=0}^{T-1} w\bigl(bc(t)\bigr) - w\bigl(bc^*(t)\bigr) = \sum_{t=0}^{T-1} w\bigl(bc(t)\bigr) - w\left(\frac{b_\sigma}{\sigma}\right) - \sum_{t=0}^{T-1} w\bigl(bc^*(t)\bigr) - w\left(\frac{b_\sigma}{\sigma}\right),$$

$$\sum_{t=0}^{T-1} w\bigl(bc(t)\bigr) - w\bigl(bc^*(t)\bigr) \geqslant M - \sum_{t=0}^{T-1} w\bigl(bc^*(t)\bigr) - w\left(\frac{b_\sigma}{\sigma}\right).$$

To obtain a contradiction suppose that $\{x^*(t)\}$ is bad, letting $T \to \infty$ we get

$$0 \geqslant \liminf_T \sum_{t=0}^{T-1} w\bigl(bc(t)\bigr) - w\bigl(bc^*(t)\bigr) \geqslant M - (-\infty)$$

which is obviously absurd. □

Next, we present a well-known result in a more general framework.[30] Brock's proof is not directly applicable since he utilizes the fact that convergence of the felicity function implies convergence of their arguments, and the benefit function used here is defined on the harvested timber levels rather than on the forest configurations. The proof presented by Mitra–Wan is based on the concavity and differentiability of the benefit (felicity) function w and cannot be used in our context, either.[31] We present an alternative proof that relies on the linearity of the function $\lambda(x(t), x(t+1))$, and can easily be extended to a concave $\lambda(\cdot,\cdot)$.

Lemma 6.2. *The Cesàro means of every good program* $\{x(t)\}$ *converge to the golden-rule forest configuration*; *namely,*

$$\bar{x}(t) = \frac{x(0) + \cdots + x(t-1)}{t} \to \hat{x} \quad \textit{when } t \to \infty. \tag{13}$$

Proof. We first observe that the convexity of Ω implies

$$\bigl(\bar{x}(t), \bar{x}'(t)\bigr) = \left(\frac{x(0) + \cdots + x(t-1)}{t}, \frac{x(1) + \cdots + x(t)}{t}\right) \in \Omega.$$

[30] This result is given in [6, Lemma 4] and in [36, Lemma 4.3]. Also see footnote 22 in this connection.

[31] Furthermore, the proof of Lemma 4.3 in [36] relies on their Lemma 3.1, whose proof is phrased in terms of the differentiability of the benefit function; see their Eq. (3.7).

Let $\bar{x}$ be any accumulation point of $\{\bar{x}(t)\}$. It is easy to see that if $\bar{x}(t_k) \to \bar{x}$ then

$$x'(t_k) = \frac{x(1) + \cdots + x(t_k)}{t_k} = \bar{x}(t_k) + \frac{x(t_k) - x(0)}{t_k} \to \bar{x} + 0$$

hence, $(\bar{x}, \bar{x})$ is an accumulation point of $(\frac{x(0)+\cdots+x(t-1)}{t}, \frac{x(1)+\cdots+x(t)}{t})$ and in consequence

$$(\bar{x}, \bar{x}) \quad \text{belongs to the closed set } \Omega. \tag{14}$$

Furthermore, and thanks to the linearity of $\lambda(x, x')$, we get that

$$\lambda\big(\bar{x}(t), \bar{x}'(t)\big) = \frac{1}{t} \sum_{l=0}^{t-1} \lambda\big(x(t), x(t+1)\big).$$

Let $\{x(t)\}$ be a good program and $\bar{x}(t_k)$ be a subsequence converging to $\bar{x}$, we know that

$$\sum_{l=0}^{t-1} w\big(bc(l)\big) - w\left(\frac{b_\sigma}{\sigma}\right) \geqslant M \quad \text{for all } t \quad \Rightarrow \quad \lim_k \frac{1}{t_k} \sum_{l=0}^{t_k-1} w\big(bc(l)\big) \geqslant w\left(\frac{b_\sigma}{\sigma}\right).$$

Hence, we have that $(\bar{x}, \bar{x}) \in \Omega$ and

$$w\big(\lambda(\bar{x}, \bar{x})\big) = \lim_k w\big(\lambda\big(\bar{x}(t_k), \bar{x}'(t_k)\big)\big) \geqslant \lim_k \frac{1}{t_k} \sum_{l=0}^{t_k-1} w\big(bc(l)\big) \geqslant w\left(\frac{b_\sigma}{\sigma}\right),$$

implying that $\bar{x} \in S(P) = \{\hat{x}\}$. □

The existence of a maximal program is furnished by the following result.

Theorem 6.1. *Any program $\{x^*(t)\}$ from $x_0 \in \Delta$ that minimizes accumulated value-loss is maximal. Consequently, there exists a maximal program from each initial state $x_0 \in \Delta$.*

Proof. First observe that

$$\begin{aligned}
\sum_{t=0}^{T-1} w\big(bc(t)\big) - w\big(bc^*(t)\big) &= -\sum_{t=0}^{T-1} \delta\big(x(t), x(t+1)\big) - \hat{p}\big(x(T) - x_0\big) \\
&\quad + \sum_{t=0}^{T-1} \delta\big(x^*(t), x^*(t+1)\big) + \hat{p}\big(x^*(T) - x_0\big) \\
&= -\sum_{t=0}^{T-1} \delta\big(x(t), x(t+1)\big) + \sum_{t=0}^{T-1} \delta\big(x^*(t), x^*(t+1)\big) \\
&\quad + \hat{p}\big(x^*(T) - x(T)\big).
\end{aligned} \tag{15}$$

Suppose, contrary to our claim, that $\{x^*(t)\}$ is not maximal. Then, there are $\epsilon_0 > 0$, T_0 and a program $\{x(t)\}$ from x_0 such that $\sum_{t=0}^{T-1} w(bc(t)) - w(bc^*(t)) > \epsilon_0$ for all $T \geqslant T_0$. Since $\{x^*(t)\}$ is good, the alternative program $\{x(t)\}$ must also be good (otherwise, the former inequality could not be hold for all $T \geqslant T_0$).

By (15) and the minimality of $\sum \delta^*$ there is $\epsilon \in (0, \epsilon_0)$ and $T_1 \geqslant T_0$ such that for all $T \geqslant T_1$,

$$\epsilon < \hat{p}\big(x^*(T) - x(T)\big).$$

Now, we may use Lemma 6.2 to obtain

$$\epsilon < \liminf_T \hat{p}\big(\bar{x}^*(T) - \bar{x}(T)\big) = 0,$$

and a contradiction arises. Thanks to Proposition 4.3, we know that given x_0, there is always a program such that $\mu(x(0)) = \sum_{t=0}^{\infty} \delta(x(t), x(t+1))$, and hence this program is maximal. □

We see next a result that complements the theorem above.

Theorem 6.2. *If $\{x^*(t)\}$ is an optimal program from x_0 then it minimizes accumulated value-loss, i.e., $\sum_{t=0}^{\infty} \delta(x^*(t), x^*(t+1)) = \mu(x_0)$.*

Proof. Suppose, contrary to our claim, that the program $\{x^*(t)\}$ does not minimize the accumulated value-loss and let $\{x(t)\}$ be a minimizer. Hence, there exists $\epsilon_0 > 0$ such that

$$\sum_{t=0}^{\infty} \delta\big(x^*(t), x^*(t+1)\big) - \sum_{t=0}^{\infty} \delta\big(x(t), x(t+1)\big) > \epsilon_0.$$

And given $\epsilon \in (0, \epsilon_0)$ there is T_0 such that

$$\sum_{t=0}^{T-1} \delta\big(x^*(t), x^*(t+1)\big) - \sum_{t=0}^{T-1} \delta\big(x(t), x(t+1)\big) > \epsilon \quad \text{for all } T \geqslant T_0.$$

By (15) we deduce

$$\sum_{t=0}^{T-1} w\big(bc(t)\big) - w\big(bc^*(t)\big) > \epsilon + \hat{p}\big(x^*(T) - x(T)\big) \quad \text{for all } T \geqslant T_0.$$

We know as well that there is T_1 such that

$$\sum_{t=0}^{T-1} w\big(bc(t)\big) - w\big(bc^*(t)\big) < \limsup_T \sum_{t=0}^{T-1} w\big(bc(t)\big) - w\big(bc^*(t)\big) + \epsilon/2 \quad \text{for all } T \geqslant T_1.$$

From the last two inequalities we get

$$\limsup_T \sum_{t=0}^{T-1} w\big(bc(t)\big) - w\big(bc^*(t)\big) > \epsilon/2 + \hat{p}\big(x^*(T) - x(T)\big) \quad \text{for all } T \geqslant \max\{T_0, T_1\}$$

which readily implies that

$$\limsup_T \sum_{t=0}^{T-1} w\big(bc(t)\big) - w\big(bc^*(t)\big) > \epsilon/2 + \lim_T \hat{p}\big(\bar{x}^*(T) - \bar{x}(T)\big) = \epsilon/2$$

contradicting the optimality of $\{x(t)\}$. □

Our next set of results are a testimony to the power of the non-interiority condition 5.2. When it holds then maximal and optimal programs coincide and in consequence, they are also equivalent to minimal accumulated value-loss programs.

Proposition 6.1. *Assume that $V = \{\hat{x}\}$. Then every maximal program is optimal.*

Proof. Let $\{x^*(t)\}$ be a maximal program from x_0 and $\{x(t)\}$ any other program from x_0.

Consider first the case where $\{x(t)\}$ is bad: by (15) and using that $\{x^*(t)\}$ is good, we deduce that

$$\lim_T \sum_{t=0}^{T} w\big(bc(t)\big) - w\big(bc^*(t)\big) = -\infty.$$

If the alternative program is good then we know that $\lim_T \sum_{t=0}^{T} \delta(x(t), x(t+1))$ is well defined, as well as $\lim_T \sum_{t=0}^{T} \delta(x^*(t), x^*(t+1))$ and also that $\lim_T x^*(T) = \lim_T x(T) = \hat{x}$, because (5.2) holds. Then, considering (15) again and letting $T \to \infty$ we get that the limit of the right hand side is defined and hence it is the limit of the left hand side:

$$\limsup_T \sum_{t=0}^{T-1} w\big(bc(t)\big) - w\big(bc^*(t)\big) = \liminf_T \sum_{t=0}^{T-1} w\big(bc(t)\big) - w\big(bc^*(t)\big) \leqslant 0. \qquad \square$$

Corollary 6.1. *Assume that $V = \{\hat{x}\}$. Then there exists an optimal program $\{x(t)\}$ from any initial state $x_0 \in \Delta$.*

From Theorems 6.1 and 6.2 and Proposition 6.1, we directly obtain the following equivalence result. The equivalence of (ii) and (iii) for the dual-aged farm, and under the hypothesis of strict concavity of the benefit function, is available in [32, Sections 2.1 and 2.2].[32]

Theorem 6.3. *Assume that $V = \{\hat{x}\}$ and that $\{x(t)\}$ is program from x_0. Then the following are equivalent*: (i) $\{x(t)\}$ *is optimal,* (ii) $\sum_{t=0}^{\infty} \delta(x(t), x(t+1)) = \mu(x(0))$, (iii) $\{x(t)\}$ *is maximal.*

7. On finitely-maximal programs

In [31], Mitra has given an axiomatic underpinning to the following notion of an optimal program. The reader should note that for the two theorems in this section, the upper semicontinuity hypothesis on the benefit function w is strengthened to continuity.

Definition 7.1. A program $\{x^*(t)\}$ is *finitely-maximal* if for any natural number $T \geqslant 1$, and any program $\{x(t)\}$ such that $x(0) = x^*(0)$, $x(t) = x^*(t)$ for all $t \geqslant T$,

$$\sum_{t=0}^{T} w\big(bc(t)\big) - w\big(bc^*(t)\big) \leqslant 0.$$

Let us first see an easy technical lemma that is an analogue of Lemma 6.1 and is original to [31, Proposition 2]. The proof we present follows the same lines as in [31], but adapted to fit our notation and to rely on programs constructed in [22]. These programs allow the planner to move from an arbitrary forest configuration to a golden-rule configuration in σ periods, and also from a golden-rule forest configuration to an arbitrary forest configuration in n periods.

Lemma 7.1. *Every finitely-maximal program is good.*

[32] Note, in keeping with footnotes 17 and 13, optimality in [31] is our notion of maximality. Such an equivalence result is also available in [51] for the RSS model under alternative hypotheses pertaining to that model.

Proof. Let $G = -(\sigma + n + 1)w(\frac{b_\sigma}{\sigma})$. Let $\{x^*(t)\}$ be a finitely-maximal program. We can assert that for all $T \geqslant 1$ $\sum_{t=0}^{T}(w(bc^*(t)) - w(\frac{b_\sigma}{\sigma})) \geqslant G$. Suppose instead that there exists $T \geqslant 1$ such that

$$\sum_{t=0}^{T}\left(w\big(bc^*(t)\big) - w\left(\frac{b_\sigma}{\sigma}\right)\right) < G. \tag{16}$$

Since $w(\cdot)$ is a non-negative function, $T > (\sigma + n + 1)$. Now consider the following $(T + 1)$ tuple:

$$\bar{x}(t) = \begin{cases} x^*(0) & \text{if } t = 0, \\ x(t) & \text{if} t = 1, \dots, \sigma - 1, \\ \hat{x} & \text{if } t = \sigma, \sigma + 1, \dots, T - n, \\ x(t) & \text{if } t = T - (n + 1), \dots, T - 1, \\ x^*(t) & \text{if } t = T, \end{cases}$$

where $(x(1), \dots, x(\sigma - 1))$ is as constructed in the proof of [22, Proposition 5.5], and $(x(T - n), \dots, x(T))$ is as constructed in the proof of [22, Lemma 6.2].

Now define the infinite sequence $\{x(t)\}$ such that

$$x(t) = \begin{cases} \bar{x}(t) & \text{if } t = 0, \dots, T - 1, \\ x^*(t) & \text{if } t > T. \end{cases}$$

Certainly $\{x(t)\}$ is a program that starts from the same initial stock as the given finitely-maximal program, and differs from it only for the subsequent $(T - 1)$ periods. But we now obtain

$$\sum_{t=0}^{T} w\big(bc(t)\big) - w\big(bc^*(t)\big) = \sum_{t=0}^{T}\left(w\big(bc(t)\big) - w\left(\frac{b_\sigma}{\sigma}\right)\right) - \sum_{t=0}^{T}\left(w\big(bc^*(t)\big) - w\left(\frac{b_\sigma}{\sigma}\right)\right).$$

The first term on the right hand side is greater than G by virtue of the fact that $w(\cdot)$ is a non-negative function, and the second term is greater than $-G$ by virtue of (16). We thus obtain the fact that the left hand side is positive, and thereby contradict the fact that $\{x^*(t)\}$ is a finitely-maximal program. □

We see next a result that complements Theorem 6.2 above, but under the standing Assumption 5.1 which simply guarantees, by virtue of Theorem 5.1, the asymptotic convergence of good programs. The essence of the argument revolves around the programs utilized in the proof of Lemma 7.1, but with the additional refinement that they make arbitrary small value-losses if the initial and terminal forest configurations in question are arbitrarily close to the golden-rule configuration.[33]

Theorem 7.1. *Assume that $V = \{\hat{x}\}$ and that w is continuous. Then a finitely-maximal program $\{x^*(t)\}$ from x_0 minimizes accumulated value-loss, i.e., $\sum_{t=0}^{\infty}\delta(x^*(t), x^*(t + 1)) = \mu(x_0)$.*

Proof. Suppose, contrary to our claim, that the finitely-maximal program $\{x^*(t)\}$ does not minimize the accumulated value-loss and let $\{x(t)\}$ be a minimizer. Hence, there exists $\epsilon_0 > 0$ such that

33 This construction can be seen as one of the principal contributions of [22].

$$\sum_{t=0}^{\infty}\delta\big(x^*(t),x^*(t+1)\big)-\sum_{t=0}^{\infty}\delta\big(x(t),x(t+1)\big)>\epsilon_0.$$

And given $\epsilon\in(0,\epsilon_0)$ there is T_0 such that

$$\sum_{t=0}^{T-1}\delta\big(x^*(t),x^*(t+1)\big)-\sum_{t=0}^{T-1}\delta\big(x(t),x(t+1)\big)>\epsilon \quad \text{for all } T\geqslant T_0. \tag{17}$$

Now, for $(\epsilon/4)$, by [22, Proposition 5.5], there exists δ_1 such that for all $x\in\Delta$ satisfying $\|x-\hat{x}\|_\infty<\delta_1$, there exists a program $\{x^i(t)\}$ from x, and differing from $\hat{x}$ only for the first σ periods, for which $\sum_{t=0}^{\sigma-1}\delta(x^i(t),x^i(t+1))<\epsilon/4$.

Furthermore, for $(\epsilon/4)$, by [22, Lemma 6.2], there exists δ_2 such that for all $x\in\Delta$ satisfying $\|x-\hat{x}\|_\infty<\delta_2$, there exists an $(n+1)$-program from $\hat{x}$, $\{\bar{x}(t)\}_{t=0}^{n+1}=(\hat{x},\dots,x)$, such that $\sum_{t=0}^{n}\delta(\bar{x}(t),\bar{x}(t+1))<(\epsilon/4)$.

For our next two claims, we shall use the non-interiority condition 5.2 to ensure, by virtue of Theorem 5.2, the asymptotic convergence of all good programs.

Since every minimal value-loss program is good by virtue of Proposition 4.2, there exists T_1 such that $\|x(t)-\hat{x}\|_\infty<\delta_1$ for all $T\geqslant T_1$.

Since the finitely-maximal program $\{x^*(t)\}$ is a good program by virtue of Lemma 7.1, there exists T_2 such that $\|x^*(t)-\hat{x}\|_\infty<\delta_2$ for all $T\geqslant T_2$.

Now let $T_3=\max\{T_0,T_1,T_2\}$, and define the infinite sequence $\{x^m(t)\}$ such that

$$x^m(t)=\begin{cases} x(t) & \text{if } t=0,\dots,T_3,\\ x^i(t) & \text{for } t=T_3+1,\dots,T_3+\sigma,\\ \bar{x}(t) & \text{for } t=T_3+\sigma+1,\dots,T_3+\sigma+n-1,\\ x^*(t) & \text{for all } t\geqslant T_3+\sigma+n.\end{cases}$$

Now, on re-writing (17) with $T=(T_3+1)$, we obtain

$$\sum_{t=0}^{T_3}\delta\big(x^*(t),x^*(t+1)\big)-\sum_{t=0}^{T_3}\delta\big(x(t),x(t+1)\big)>\epsilon.$$

Since $\delta(x^*(t),x^*(t+1))$ is non-negative for all t, we obtain

$$\sum_{t=0}^{T_4}\delta\big(x^*(t),x^*(t+1)\big)-\sum_{t=0}^{T_3}\delta\big(x(t),x(t+1)\big)>\epsilon \quad \text{where } T_4=(T_3+\sigma+n). \tag{18}$$

Finally, since

$$\sum_{t=T_3+1}^{T_4}\delta\big(x^m(t),x^m(t+1)\big)<(\epsilon/2),$$

we obtain from (18),

$$\sum_{t=0}^{T_4}\delta\big(x^*(t),x^*(t+1)\big)-\sum_{t=0}^{T_3}\delta\big(x(t),x(t+1)\big)-\sum_{t=T_3+1}^{T_4}\delta\big(x^m(t),x^m(t+1)\big)>(\epsilon/2).$$

Since the program $\{x^m(t)\}$ starts from x_0 and is identical to the program $\{x^*(t)\}$ for all $t\geqslant T_4$, we obtain

$$\sum_{t=0}^{T_4} w\big(bc^m(t)\big) - w\big(bc^*(t)\big) > (\epsilon/2).$$

But this contradicts the assertion that $\{x^*(t)\}$ is a finitely-maximal program. □

We can now turn to the unification that has motivated this essay. Theorem 7.1 allows us to strengthen Theorem 6.3 into the following equivalence result.

Theorem 7.2. *Assume that $V = \{\hat{x}\}$, that w is continuous, and that $\{x(t)\}$ is program from x_0. Then the following are equivalent*: (i) *$\{x(t)\}$ is optimal,* (ii) $\sum_{t=0}^{\infty} \delta(x(t), x(t+1)) = \mu(x(0))$, (iii) *$\{x(t)\}$ is maximal,* (iv) *$\{x(t)\}$ is finitely-maximal.*

The equivalence of (iii) and (iv), under the hypothesis of strict concavity of the benefit function, is available in [31, Theorem 7].[34] In the introduction, we have already remarked on the need for a different method of proof in the setting we are working in. The reader is also invited to compare Theorem 2.2 and the analogous results in [54].

8. Concluding remarks

The non-interiority condition presented in this paper has served as a synthesizing criterion for the analysis of the Mitra–Wan tree farm by ensuring the asymptotic convergence of good programs, and by being both a necessary *and* sufficient condition for this convergence when the benefit function is concave. Premised on such an asymptotic convergence, a theory of undiscounted dynamic programming is developed for a general intertemporal model in Dana and Le Van [11,12], for the 2-sector RSS model in Khan and Mitra [21], and for the dual-aged forest in [32]. All of these results are set in the context of strictly concave felicity (benefit) functions, and there is little doubt that an analogue of the non-interiority condition reported here will also serve to move this theory towards completion by an extension to the larger class of functions isolated here. Indeed, the results developed can be extended in a very direct way to any model where the benefit is a function of the consumption, and the consumption can be represented as a concave function of the pair $(x, x') \in \Omega$. The only point of difficulty in such an extension is an adequate characterization of the set V, and a linking of this set with the set of zeroes of the discrepancy function S_f, a linking that would revolve on the particularities of the capital theoretic model being investigated and extended. The rest of the development would simply follow the lines of the analysis presented in Section 5. In this connection, we also leave as an open problem the identification of a necessary and sufficient analogue that takes the general theory of intertemporal resource allocation, developed and reported in [25–27] to such a larger class of felicity functions. A theory generalized along both of these lines should be of especial use in considerations that go beyond the deterministic to the stochastic context.[35]

It is worth reiterating that our focus in this paper, following the original conception of Mitra–Wan [36], has been on infinite horizon optimal programs without discounting, which is to say, on the undiscounted long-run. We already know from [35,13] and [47] the difficulties of delineating

[34] Note, in keeping with footnotes 17 and 13, optimality in [31] is our notion of maximality. Note also footnote 15.

[35] See [9] and [28] where the growth of a tree is modeled as a Weiner process. A generalization of the Mitra–Wan forestry model to more standard Brock–Mirman type models in the growth and uncertainty literature also remains to be accomplished.

optimal transition dynamics in the discounted setting, and whereas a full understanding of the model can hardly be had without a resolution of these difficulties, substantial analysis of the undiscounted case that is feasible remains to be done. More importantly, and given the recent emphasis on numerical computation,[36] the proximity of finite horizon optimal programs to their infinite horizon counterparts (as is investigated in [23] for the RSS model and [22] for the Mitra–Wan model), and questions of their sensitivity to initial and terminal stocks (as investigated in [7] and [29]), remain to be investigated in the setting considered in this paper. We leave such investigations for future work.[37]

Indeed, the undiscounted setting has special reference to environmental economics: discounting by a planner makes even less sense when issues of climate change are in question or the value of a forest goes beyond its timber yield and takes environmental well-being into account.[38] This leads to a situation where the stock variables are arguments in the benefit functions, as in [38, 39]. Indeed, as emphasized in [8, p. 453] and [18] in the context of capital theory, and by [2,3,44] more generally, the subject naturally leads into intergenerational and intertemporal equity issues. Overlapping generations, with each succeeding generation having sole property rights to the forest, will lead to a complementary conception in which, unlike the perishability of Samuelson's chocolates, the commodity has durability over a finite number of generations.[39] It is certainly of interest to see how the arguments developed in this paper fare for such a setting, and this too, we leave for future work.

But moving beyond a general theory of undiscounted dynamic programming, and beyond a general statement regarding applications to deterministic and stochastic discrete-time dynamic systems in capital theory and environmental economics, the relevance of the non-interiority condition presented in this paper, and specifically of Theorem 5.2, is clear for a generalization of the non-interiority condition presented in this paper, and specifically Theorem 5.2, can be used to generalize the existence and asymptotic results for maximal and optimal programs in the RSS model presented in [20,51].[40] It is clear that the non-interiority condition will also serve as a synthesizing criterion for contexts (other than the Mitra–Wan forestry model) cataloged in [30]. We leave these verifications as exercises for the interested reader.

Appendix A

In [31], for any concave function $u : X \to \mathbb{R}$, with X a convex subset of a linear space, Mitra has defined a strict mid-concave function as one where for x_0, x_1 in X, and $x_0 \neq x_1$,

$$u(x_{\frac{1}{2}}) > \frac{1}{2}u(x_0) + \frac{1}{2}u(x_1). \tag{19}$$

We make the simple observation that this is tantamount to the assumption of strict concavity of u. Let $x_\lambda = \lambda x_0 + (1 - \lambda)x_1$. We need to show that

$$u(x_\lambda) > \lambda u(x_0) + (1 - \lambda)u(x_1) \quad \text{for all } \lambda \in (0, 1). \tag{20}$$

[36] See the textbook of Judd [17] and his references.

[37] Arkin and Evstegneev [1] and Zaslavski [52] are comprehensive references to this set of issues.

[38] See Brock [8] for an emphasis on the undiscounted setting in the context of environmental and resource economics.

[39] For references and recent work on the OLG model, see [50] and [24], and the textbook [4].

[40] Since this paper was written, the authors have studied this problem in *Set-Valued and Variational Analysis* 19 (2011) 135–156.

We consider first the case where $\lambda \in (0, \frac{1}{2})$. We can write x_λ as the convex combination of $x_{\frac{1}{2}}$ and x_1,

$$x_\lambda = \lambda x_0 + (1-\lambda)x_1 = \lambda(x_0 + x_1) + (1-2\lambda)x_1 = 2\lambda x_{\frac{1}{2}} + (1-2\lambda)x_1$$

Now, the concavity of u assures that $u(x_\lambda) \geqslant 2\lambda u(x_{\frac{1}{2}}) + (1-2\lambda)u(x_1)$ and thanks to (19) we know that

$$\begin{aligned} u(x_\lambda) &\geqslant 2\lambda u(x_{\frac{1}{2}}) + (1-2\lambda)u(x_1) \\ &> 2\lambda\left(\frac{1}{2}u(x_0) + \frac{1}{2}u(x_1)\right) + (1-2\lambda)u(x_1) \\ &= \lambda u(x_0) + (1-\lambda)u(x_1) \end{aligned}$$

proving (20) for $\lambda \in (0, \frac{1}{2})$.

Next, to deal with the case where $\lambda \in (\frac{1}{2}, 1)$, we write x_λ as the convex combination of x_0 and $x_{\frac{1}{2}}$: $x_\lambda = (2\lambda - 1)x_0 + (1 - (2\lambda - 1))x_{\frac{1}{2}}$. And repeating the steps above, we prove (20) for $\lambda \in (\frac{1}{2}, 1)$.

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