Turnpike theory: a current perspective

This 2012 perspective of the 1987 *Palgrave* entry on 'turnpike theory' highlights the subsequent development of the subject in the light of a critical re-reading of the original. It distinguishes the 1949 conception, a response of Samuelson to a 1945 von Neumann challenge to the reception of his growth model in the economic literature, from the more capacious 1976 outline furnished by McKenzie. Thus, it differentiates asymptotic convergence of infinite-horizon optimal programs from what it terms their finite-horizon, classical turnpike counterparts. It identifies a move from the investigation of general theorems to a more detailed working of simple examples, and reports results on specific models of 'choice of technique' in development planning, and of lumber extraction in the economics of forestry. Drawing on ongoing advances in the field of dynamical systems, it sees such models as both litmus tests of the general theory and as productive settings to study the *rationalisability* of policy functions and a 'folk theorem of intertemporal resource allocation'. The entry concludes with brief speculative remarks for future directions.

It is common knowledge within the economics profession that the substance underlying the term *turnpike*, and the theorem to which this noun served as an adjective, entered economic theory as a conjecture in a 1949 Rand memorandum authored by Paul Samuelson. Referring to the optimal rate of growth λ^* in the 1935–36 von Neumann growth model, and to the maximal, balanced factor proportions υ^* that underlie it, Samuelson was to write:

> Let us return to the interpretation of the optimal rate of growth λ^*. Growth in the equilibrium mode will ultimately surpass any other rate of balanced growth. This suggests that if we start with any factor proportion $\upsilon \neq \upsilon^*$, it will still pay us if we are investing *for the very far future* to get into (or near) the equilibrium mode. At worst, we can do this by throwing away some of whichever factor is initially redundant as compared to υ^*; and at best we can obviously make some use of the redundant factor. [C]learly, at first υ would not be near υ^* but would ease in gently toward υ^*. And it is also clear that if we prescribe υ_0 and want the maximum υ_n, then as we finally get near *n*, it will pay to leave υ^* even if we are already there or near there. One would conjecture, therefore, that beginning with $\upsilon_0 \neq \upsilon^*$ and ending with $\upsilon_n \neq \upsilon^*$, the optimal time path of υ would look something like Figure 14 for large *n*. As *n* gets large, the average υ should approach υ^* (S33:489).
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> [In all references to Samuelson (and Debreu and Koopmans) from (Samuelson, 2008) (and (Debreu, 1983) and (Koopmans, 1970) respectively) the first number indicates the chapter, and the second, the page within it.]

It is also well-known, at least since Lionel McKenzie gave it prominence in his 1976 Fisher–Schultz lecture, that the term itself, as opposed to its underlying substance, occurs in Chapter 12 of a 1958 volume authored by Dorfman, Samuelson and Solow (Dorfman, Solow and Samuelson, 1958) (henceforth DOSSO). One may quote from McKenzie's quotation from DOSSO:

> It is exactly like a turnpike paralleled by a network of minor roads. There is a fastest route between any two points; and if the origin and destination are close together and far from the turnpike, the best route

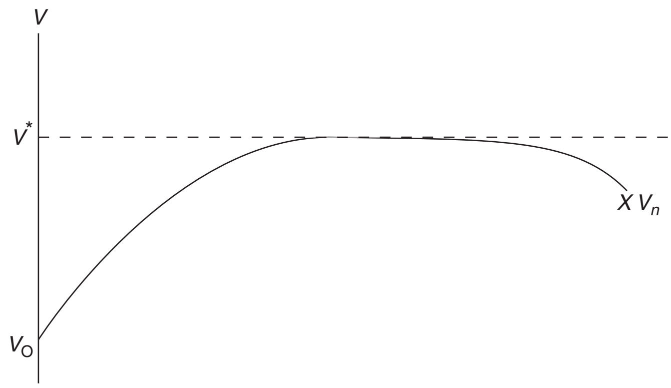

Figure 1 Figure 14 in Samuelson (1949).

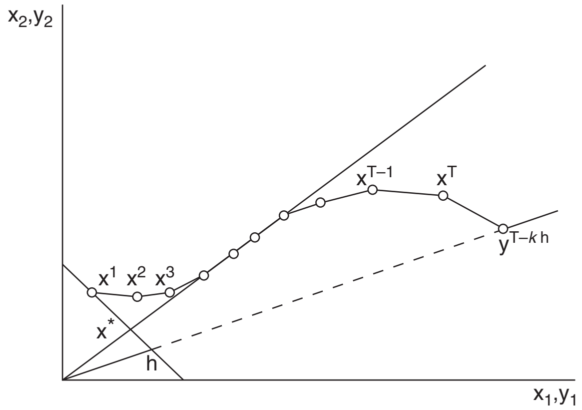

Figure 2 Figure IX in Koopmans (1964).

> may not touch the turnpike. But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end (841).

[All references to McKenzie's papers are from (McKenzie, 1986); this is also the relevant source for papers prior to 1986 not referenced. A number on its own at a quotation's end refers to the page number in the relevant reference.]

McKenzie was to conclude the first paragraph of his lecture with the statement that 'It is due to this reference, I believe, that theorems on asymptotic properties of efficient, or optimal, paths of capital accumulation came to be known as *turnpike theorems*.'

In terms of a time-line pertaining to *turnpike theory*, the benchmark dates 1960 and 1965–66 follow 1949 and 1958. In a 1960 chapter (S26), an appendix originally written for possible inclusion in DOSSO, Samuelson returns to the subject under the heading 'efficient paths of capital accumulation in terms of the calculus of variations'. After a reference to the discrete-time treatment in DOSSO, an analogy to the von Neumann growth rate, and a consequent continuous-time conjecture, is adduced:

> For the discrete-time case, the so-called 'turnpike theorem' was enunciated. This asserts that if the goal of maximization is far enough ahead, one will want to travel very near to the von Neumann mode of balanced growth. The economic common sense of this evident, but it

> would be well to have a general proof for the continuous-time case. The remaining two sections will deal with this and related matters. Economic intuition suggests that the local result proved here must also hold in the large, but this must remain an open question (S26:297).

In addition to its uplifting of maximally balanced growth programs to intertemporally efficient ones, the chapter emphasises two other novel points of view: it (i) reads the theorem as part of the 'formal mathematical analogy between classical thermodynamics and mathematic economic systems' (more fully explored in another contemporaneous chapter (S44)), and it (ii) identifies the catenary property as its local manifestation. The classical Euler–Lagrange first-order conditions of the classical calculus of variations, as in the modernised formulation of Carathéodory and Poincaré, when linearised around the singular point, furnish real eigenvalues that come in 'pairs of opposite signs... that lead to the desired catenary motions around the saddle point'. By 1965, the earlier uplifting to four notions of efficiency (a re-formalisation, already available in DOSSO, of the 'best route') had been uplifted further to the 1928 Ramsey maximand of an integral of felicities depending on consumption, and with the 'usual 'small vibrations'' analysis of the motions in the neighborhood of the equilibrium point' correspondingly reformulated. However in this, Samuelson (S136; also Figure 3 and Footnote 8) is in company with Atsumi, Koopmans and Cass, and he will pursue this matter for at least the next five years; see S137–S142, S150, S223–224, K26–K28 (Figure 10 on pp. 575, 591 (Vol. 1) and Figure 5, p. 176 (Vol. 2)) and for other references. In any case, by 1966, the original version of the theorem, based on the maximisation of terminal stocks, had also been understood as a global result in a flowering in which Kuhn, Hicks and Morishima in 1961, Radner and Furuya-Inada in 1962, McKenzie in 1963, and Nikaido, Inada and Koopmans in 1964 played an essential part; see S136 (also Footnote 1) and K23 (Footnotes 4–9 and 1–2) for these references.

What this abbreviated narrative omits is that the turnpike conjecture, and the substantial theorems in which it found, and continues to find, its rigorous crystallisation, originated as a local dispute between Samuelson and von Neumann concerning framing: a missed opportunity 'sometime in 1945' for Samuelson to respond to von Neumann's challenge that 'his model of general equilibrium... involved new kinds of mathematics which had no relation to the conventional mathematics of physics and maximization' (S130:15); also see S406:75. This is surely not common knowledge within the profession, and despite McKenzie's masterful 1986–1987 surveys, not as well-appreciated even by the *cognoscenti* as it ought; see (McKenzie, 1986, McKenzie, 1987). In his 1998 Richard Ely lecture, McKenzie (McKenzie, 1998) himself recounts the episode, but takes an altogether rosier view of the outcome than Samuelson himself (a full cigar versus half a cigar), and represents Samuelson's response to von Neumann's claim that 'maximization of an objective function had no part in his theory' as saying that 'maximization would enter once disequilibria were considered'. What was new in von Neumann's 'general equilibrium theory' was of course the notion of a saddle point and the minimax theorem; and as Karlin's 1960 text was to show most transparently, this entailed a separation: the question of the existence of a maximal balanced growth program treated independently of its price characterisation involving a minimal interest rate. Thus, in an ironic twist, the von Neumann use of the fixed-point theorem was not to be brought into play at all in the proof of his theorem; see the polished treatment of the minimax theorem in (Simons, 2008, Chapter 1). But Negishi notwithstanding (see (McKenzie, 2002)), this separation between existence and characterisation of equilibrium, with fixed-point theory servicing the first and

Hahn–Banach theory the second, and neither concerned with computation, was to leave its footprint on the development of Walrasian general equilibrium theory in the sixth decade of 20^{th}-century economics; along with DOSSO and the 1971 Arrow–Hahn (Arrow and Hahn, 1971) reworking of Debreu's 1959 classic, see (Khan, 2010). Von Neumann's challenge to Samuelson was then to provide an asymptotic implementation of his theorem on 'good' infinite programs using finite-horizon ones that were optimal in the sense conventional at the time. In another, more current, vernacular, this was to ask for a microeconomic foundation, based on optimisation, to the macroeconomic regularities of the maximal–minimal growth–interest pair that von Neumann had uncovered. Such a microfoundation was subsequently supplied by Arrow, Debreu and Scarf (D1, D5, D11 and references) in what can now be seen as the second and third fundamental theorems of welfare economics: to continue using the services of the separating hyperplane theorem to exhibit Pareto optimal and core allocations as Walrasian (price) equilibria, one with redistribution and the other without. But since general competitive analysis was to repress infinite-dimensional commodity spaces, which is to say, repress the 1953–54 analyses of Malinvaud (D5:104) and Debreu (D5), and the 1958 treatments of Hurwicz (Arrow, Hurwicz and Uzawa, 1958) and Samuelson (S21), and to develop, until 1971 at any rate, in the setting of a finite number of commodities, there was no general equilibrium ground on which turnpike theory could turn and find play. It consequently receded and moved away, von Neumann's contribution bifurcated; however, see (McKenzie, 2002, Bewley, 2007) as attempts at restitution.

The two arms of this bifurcation were of course a theory of resource allocation premised on heterogeneous agents, a heterogeneity with a cardinality of a finite set or that of a continuum, but with a finite number of commodities; and another premised on a representative agent, or, to coin a phrase, a continuum of representative agents, but with a *bona fide* infinite-dimensional commodity space. And from 1965–66 onwards, turnpike theory was to thrive in the context of the latter. The 1965 Atsumi-von-Weiszäcker reformulation of Ramsey's criterion led rather quickly in 1967–70, at the hands of Gale, McKenzie and Brock, to a complete treatment of the theory of the existence of optimal programs in Ramsey-like situations when the aggregate is not well defined. However, unlike Brock and Gale, McKenzie retained his primary focus on the turnpike theorem, and on its generalisation to a fully multi-sectoral environment. This focus is total, and the first two theorems of his paper surely mark 1968 as an important benchmark date for turnpike theory. As he was to write later in the *Palgrave* (McKenzie, 1987):

> The spirit of the original turnpike theorem is not well preserved in the aggregative model since the emphasis in the original theorem lies in the relative composition of the capital stock. Turnpike theorems for the general multi-sectoral model with a Ramsey objective and a von Neumann technology were first proved by Gale (1967) and McKenzie, (1968). Their order of proof does not differ from that of Atsumi, which is, in turn, parallel to the proof used by Radner in the model with maximal growth as an objective.

McKenzie sketches the outline of his proof in (McKenzie, 1987), and again in his 2002 book (McKenzie, 2002). As in (Karlin, 1960), one draws on the separating hyperplane theorem to associate a price system to the facet of stationary programs, maximal in the obvious sense, and once this is done, to derive bounds when the finite-horizon optimal program is not 'on or near' the facet of stationary programs. And to be sure, leaving finite-horizon programs aside, and on appealing to the Atsumi-von-Weiszäcker optimality criterion, one also obtains the existence of such infinite-horizon programs. If the stationary program is unique, a sharper existence result is obtainable. In any case, what is clear is that by 1968, *good* and *bad* programs, the *von Neumann facet* and its possible shrinkage to a singleton, as in Brock's seminal 1970 treatment, the emphasis on *bunching* or *clustering* of optimal programs, as opposed to convergence to a unique equilibrium, had all became staples in the vernacular of turnpike theory. [All references in this paragraph are available in (McKenzie, 1986; McKenzie, 1987).]

The year 1976 takes its place alongside 1949, 1958, 1965–68 as a benchmark year for turnpike theory. In his 1976 Fisher–Schultz lecture, already referred to above, McKenzie surveys where the matter stood, and observes that 'there were no global results for perhaps the most relevant case for decision making, the maximization of a discounted sum of utility over time with scarce labor, [but that] in the past two years the situation has changed significantly' (843). He cites Scheinkman for a first attempt at what will later become the 'neigborhood turnpike theorem', Rockafellar and Cass and Shell for conditions on the concavity of the felicity function, Brock and Scheinkman for a rendering in continuous-time, Araujo and Scheinkman for a dominant diagonal condition which 'does not translate directly into the degree of concavity or the size of the discount factor', and his own result concerning non-stationary felicities; see (Mitra, 2005) for a more recent synthesis. It is important to understand the need for these additional conditions in the discounted case: proofs in the undiscounted case are based heavily on the fact that the value-loss of good programs must converge to zero, thereby guaranteeing their convergence to the von Neumann facet. This is no longer true in the discounted case, as also brought out in (McKenzie, 1987, p. 715). But there is another less positive reason for singling out 1976. It is the year when turnpike theory, deprived of its moorings in general equilibrium theory, over-reaches. McKenzie puts forward a tri-partite classification of turnpike theory – the middle, the early and the late turnpike – and with this classification, turns a local challenge regarding asymptotic implementation of an infinite-horizon program into the global study of intertemporal allocation of resources. More specifically, he widens the meaning of the word *asymptotic* to include, under the category of the late turnpike, the convergence of optimal infinite-horizon programs to stationary programs, and to their tendency to 'bunch or cluster' together in the far future. Indeed, the references cited in his 1976 lectures all pertain to the *late* turnpike. Thus, in his *Handbook* chapter a decade later, McKenzie was to write:

> The theory... will cover both discounted and undiscounted utility. We will seek to determine the asymptotic behavior of maximal paths, which display a tendency to cluster in the sufficiently distant future from whatever capital stocks they start. In models with stationary utility functions the clustering has been seen as convergence to a stationary path along which capital stocks are constant. The existence of infinite optimal paths in the stationary disaggregated model were proved by Gale (1967). Asymptotic theorems in this model were proved by Atsumi (1965), Gale (1967), and McKenzie (1968),... [and they] were extended to [discounted] models by Scheinkman (1976) and by Cass and Shell (1976). Excellent examples from the theory of competitive equilibrium are the recent works of Becker (1980), Bewley (1982) and Yano (1981), where the turnpike results... are used to prove that competitive equilibria approach stationary states over time. It has been

> suggested that our subject is best described as the study of economizing over time.

This extended quotation, with its going over of references already read and referenced, is necessary to bring out how this widening of the term *asymptotic*, and its heightened elevation to the entirety of the qualitative theory of economic dynamics, robs turnpike theory of its very identity. The *late* turnpike is not a turnpike, in the sense that it typically requires infinite time to get on it, and with no terminal capital stock there is 'no getting off it'. In the words of Koopmans (K:201, Volume 2), 'there is no fatal cut-off point', unlike the early and middle turnpike twins. In the quotation above, other than those to Atsumi and McKenzie himself, all other references pertain to the late turnpike, rather than to the *early* and *middle* ones, and by 1998, the terminology proliferates to make Ramsey, von Neumann and Samuelson serve as adjectives to the noun 'turnpike'; see (McKenzie, 1998, Section II and III). A sustained case for the rescue and protection of the original Samuelsonian conception of the theory from its more universalistic 1976 ambitions was first made by Khan and Zaslavski (Khan and Zaslavski, 2010) in 2010.

McKenzie's 1976 reading was immensely influential: it did not simply balance work on the early and middle turnpike by those on the late turnpike, but (barring a few exceptions) entirely eliminated the former from the mainstream journals, handbooks and anthologies; see (Benhabib, 1992, Majumdar, Mitra and Nishimura, 2000, Aghion and Durlauf, 2005, Dana et al., 2006). Thus, in Mitra's important 2005 synthesis of the 1976 results, it is 'clarified that it is only in the sense of 'global asymptotic stability'' that the term 'turnpike property'' is used in this paper'; also see (Khan and Piazza, 2011) for such a trajectory extending from the 1977 Araujo and Scheinkman analysis to Bewley's 2007 text. Indeed, McKenzie begins his own *Palgrave* entry with the classical and neoclassical economists, as represented by Mill and Cassel, and originates the subject in the time-honoured conception of the 'eventual convergence of the economy to a stationary state as a consequence of the growth of population and the accumulation of capital, in the absence of continual technical progress or continual expansion of natural resources'. He opens the penultimate paragraph of his entry with the summary statement 'The theorems that have been reviewed are all concerned with the convergence of optimal paths to stationary optimal paths' – a statement that applies even more so to the references in the concluding paragraph. In what is surely a classic, divided into five sections – turnpike theorems for the von Neumann model, for the Ramsey model, for Ramsey models with discounting, for competitive equilibria, and for generalisations to habit formation uncertainty and non-convex technologies – only the first concerns the early and middle turnpike, and the hazard of an opening getting reified into a doctrinal closing was unfortunately realised; see (Khan and Piazza, 2011), and especially their introduction, for a substantiation of this reading. The point of course is that it is the classical conception that was novel to both economic theory and to applied mathematics, and it was not a re-packaging of the qualitative theory of differential and difference equations, albeit ones generated by the Euler–Lagrange conditions of the calculus of variations or their discrete-time counterparts. Indeed, even though Radner's value-loss methods play a role in the important work of Araujo and Scheinkman on the asymptotic convergence of optimal infinite-horizon trajectories, it is the implicit function and Hirsch–Pugh theorems, time-tested stalwarts of dynamical-systems theory, that bear the brunt of the lifting; (Carlson, Haurie and Leizarowitz, 1991, Zaslavski, 2005) may be the only mathematical writers to make and appreciate the distinction.

With this 2012 re-reading of a 1987 reading behind us, we can turn to subsequent developments and ask whether there is very much to say, if anything, about turnpike theory in the classic Samuelsonian sense. However, prior to this, we need to highlight 1986 as another benchmark date for turnpike theory. It is in that year that Boldrin, Deneckere, Montrucchio and Pelikan (henceforth BDMP) investigate the relevance of the 1972–1974 Sonnenschein–Mantel–Debreu theorems (D16 with its five references) for the theory of optimal intertemporal allocation of resources; see (Boldrin and Montrucchio, 1986a, Boldrin and Montrucchio, 1986b, Deneckere and Pelikan, 1986) and reference to antecedent work by Montrucchio (1984). They show that any twice continuously differentiable function can be *rationalised* as the policy function of an appropriately defined dynamic optimisation model, and taking their cue from May (1976) and the twice-differentiability of the logistic map, they conclude that anything, including chaotic trajectories, can emerge as a result of optimisation by a representative agent over infinite time. The implications of this work for turnpike theory (middle, early or late) could hardly be minimised; see (Dana et al., 2006, Chapters 4 and 6) and (Khan and Piazza, 2011) for an outline of the theory as it has taken shape since 1986, and for its reliance on Sharkovsky's theorem. However, the point is that this work has different implications depending on which of the two conceptions of the subject is presupposed. If one is focused on the late turnpike, and on the asymptotic convergence of optimal trajectories to the benchmarks of a stationary or a quasi-stationary model (the terms are McKenzie's (McKenzie, 1986)), it surely delivers a resounding, if not fatal, refutation to the turnpike conjecture. If, on the other hand, one is focused on the early or middle turnpikes, and on the approximation of infinite- by finite-horizon programs with a specified terminal capital stock, the turnpike conjecture remains very much alive and viable. If confronted with an optimally chaotic solution to infinite-horizon problem, it can still ask whether an optimal solution to a finite-horizon problem, for a large enough horizon reflecting the tolerable error of approximation, stays close to the infinite one – partakes and participates in the chaos of its parent, so to speak. Samuelson's 1976 periodic turnpike theorem (S224) may already be a pioneering answer to a question yet to be formulated and asked, the only (not inessential) difference being that in his case the periodicity is exogenous rather than endogenous. Indeed, putting the question this way then leads to giving a constructive twist to what may have so far seemed to be a purely semantic issue, a rather negative attempt to distinguish, and dissociate, Samuelson from McKenzie. This is then to ask whether there is a relationship between their two differing conceptions of turnpike theory: the local versus the global – local in the sense of capital theory being a locality of the general environs of intertemporal resource allocation. Put more sharply, can a result on the asymptotic convergence deliver a classical turnpike theorem as its corollary?; and conversely, can a classical turnpike theorem imply, in the limit, as the time-horizon is extended without bound, a coming-together of the optimal trajectories? For initial exploratory studies, see (Khan and Zaslavski, 2010), and subsequent to that work, (Khan and Piazza, 2011, Zaslavski, 2009, Zaslavski, 2010).

However, even before the BDMP floodgates opened, Scheinkman's 1976 question as to the extent to which global asymptotic stability of optimal infinite-horizon programs could be salvaged from an undiscounted to a discounted setting, hovered over the subject. In 1983, McKenzie writes:

> Asymptotic theory for optimal paths of capital accumulation is more difficult when the utility function for the single period is concave, but not strictly concave. However, in the case of stationary models where future utility is not discounted, the theory is rather fully developed in McKenzie (1968, 1976). In the case of discounted utility and quasi-stationary models this order of proof does not succeed, because convergence of optimal paths to the facets on which optimal stationary paths lie cannot be proved to be asymptotic on the basis of arguments from value losses, or utility gains. In order to carry the argument further, we must use the convergence of the von Neumann facets associated with discount factors to the von Neumann facet of the undiscounted model as the discount factor approaches unity (330–331).
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> [Under the adopted bibliographic convention, McKenzie (1983) is available in (McKenzie, 1987).]

This is the genesis of what has come to be called McKenzie's *neighborhood turnpike* theorem: 'the larger the neighborhood chosen, the smaller the discount factor allowed'. The theorem was answered, potentially negatively, but more than a decade later. In 1995, Nishimura and Yano were to show that for any discount factor arbitrarily close to unity, one could construct a two-sector LS (Leontief–Shinkai) model whose optimal policy function would exhibit ergodically chaotic (not simply topologically chaotic) optimal trajectories. The question of course is how compelling this answer was: McKenzie speaks of a *given* model and Nishimura and Yano respond in terms of a *constructed* model. In any case, the following year the profession was to be presented with even more remarkable results: Mitra and Nishimura and Yano were to discover, entirely independently, 'exact discount factor restrictions for dynamic optimization models'. Loosely speaking, this was to identify, in the context of a general model, a threshold for the discount factor that was necessary and sufficient for an optimal trajectory in that model to exhibit period-three cycles, and therefore by Sharkovsky's theorem, cycles of all periods; for details and references, see (Majumdar, Mitra and Nishimura, 2000, Chapters 11, 12) and (Dana et al., 2006, Chapter 4). It was then only natural that within a decade of these achievements, the three introductory paragraphs of McKenzie (1983) would be consummated in a precise 'folk-theorem of optimal accumulation'. The theorem was to announce, in the context of a general model, the potential existence of a threshold discount factor, such that optimal trajectories would exhibit chaotic behaviour for all factors less than that threshold, and asymptotic convergence, which is to say, a late turnpike for those above it; see (Khan, 2005) for an initial formulation. And not unlike its game-theoretic cousin, this folk theorem, even in its precise formulation, still remains a conjecture for a general-enough class of models; see (Khan and Mitra, 2010, Khan and Mitra, 2011, Khan and Mitra, 2012).

The BDMP work, and subsequent papers that gave it grounding, had, by necessity, considered simple and specific, typically two-sector, production structures. As an unintended consequence, this then changed the tone of the subject: it retreated from its aspiration towards increasing generality – in addition to non-convexity and uncertainty, even an embrace of non-stationarity – to one looking towards the explicit and detailed working of specific examples. In this, it was also spurred on by the theory of dynamical systems, and more specifically, 'iterated functional systems (IFS)'; see Barnsley (Barnsley, 2006) and his references, and for their influence on economic theory, see (Bhattacharya and Majumdar, 2004, Bhattacharya and

Majumdar, 2007) for references to papers of Bhattacharya, Majumdar, Mitra and others. In particular, the aggregative model popularised by Ramsey, and Solow in 1956 (currently the RCK [Ramsey–Cass–Koopmans] workhorse in macroeconomics) and the two-sector versions swirling around the LS (Leontief–Shinkai) model already mentioned in connection with the work of Nishimura and Yano, and investigated further in Fujio (Fujio, 2009), were to be complemented by two settings originating in subjects not apparently related to each other, and both neglected only until recently. The first dates to the early 1960s, when it served as the lightning rod for polemics between two transatlantic locations, and was dusted off and rediscovered as the so-called RSS (Robinson–Solow–Srinivasan) model by Khan and Mitra (2005); see (Khan and Mitra, 2005) for details as to genealogy and references. The second is also a re-visiting, but this time of Samuelson's work on the economics of forestry (S218) as the MW (Mitra–Wan) model; see (Khan, 2005, Khan and Piazza, 2012) for references to the pioneering papers. Both involve a period-by-period allocation of an inelastically supplied resource: in the first case, labour to produce one or more types of machine chosen from a finite set, or a consumption good along with any of the machines that may be available; and in the second case, land partitioned out among a finite set of tree-vintages, all costlessly grown. Machines depreciate at the same rate, trees grow and yield lumber at different rates. The objective is to maximise an undiscounted, or discounted, stream of the consumption good, with or without strictly concave, or even concave, felicities; see (Khan and Zaslavski, 2009, Khan and Piazza, 2011, Khan and Piazza, 2012). Thus both settings are multi-sectoral, and thereby immune to McKenzie's complaint regarding the RCK model as not in keeping with the original spirit of turnpike theory. The point, however, is that, simple as these two settings are, they are not simple enough! On the one hand they have led to a transparent geometric consolidation, and considerable sharpening, of the theory, but on the other hand, the dynamics they exhibit are rich enough to defy full understanding even for special cases of a single type of machine or a dual-aged forest. Work is ongoing, but has already yielded insights for capital theory in the large that go beyond the two instances themselves; for existence and asymptotic convergence, rationalisability, chaotic dynamics, parametric restrictions, policy correspondences and several bifurcations revealing an unexpected intricacy to the 'folk theorem', see respectively (Khan and Zaslavski, 2009, Khan and Piazza, 2010, Khan and Piazza, 2010, Khan and Piazza, 2012, Khan and Piazza, 2011, Khan and Mitra, 2012, Khan and Mitra, 2011, Khan and Mitra, 2012, Khan and Mitra, 2010) and their references.

The above two paragraphs concern asymptotic convergence and the late turnpike: the question concerning progress on the classical turnpike theorem, asked above and left hanging, hangs still. It is of interest that it is the early work on non-classical environments, non-convex technologies and uncertainty, that scrupulously maintained a distinction between the two conceptions and preserved the autonomy of the original. The 1982 analysis of Majumdar and Nermuth (Majumdar and Nermuth, 1982) in the one case, and the 1998 analysis of Joshi (Joshi, 1998), building on the pioneering 1978 connection to the martingale property in Follmer and Majumdar (Follmer and Majumdar, 1978), on the other, is surely worth careful study even today; also see (Bhattacharya and Majumdar, 2007, Arkin and Evstigneev, 1987) and their references. More recently, even in a deterministic, non-stochastic context, there has been a breaking of new ground in the context of the RSS and MW models delineated above. This concerns a further weakening of the optimality notion. Thus, rather than the Samuelsonian triple limit (S130:15), an interesting 'quarter limit' seems to be involved in the four separate considerations that are quantified. In other words, for any given levels of the initial and terminal capital stocks, υ_0 and υ_1, and for any three given levels of approximation, ε_1, ε_2 and ε_3, one can find a large but finite time horizon $T(\upsilon_0, \upsilon_1, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ such that any ε_3-optimal program starting from υ_0 and required to furnish υ_1 at its termination, spends $(1 - \varepsilon_1)$-proportion of the time in an ε_2-proximity to the turnpike for all time periods that extend over T. Such results were proved for the RSS model in (Khan and Zaslavski, 2010), and for the MW forestry model in (Khan and Piazza, 2011). In each case, the theorems are shown to yield *uniform* asymptotic convergence of the maximal stationary programs and thereby generalise corresponding results in (Khan and Zaslavski, 2006, Khan and Piazza, 2010). With so many epsilons, surely *nonstandard analysis* waits in the wings; it may also be the right mathematical language to express McKenzie's notions of 'bunching and clustering' in the context of the classical Samuelsonian conjecture, and make the first steps towards a set-valued turnpike theory. To return to a theme broached earlier, the exploratory analysis presented in (Khan and Piazza, 2011) bunches the full continuum of periodic programs together, and investigates the entire set as a possible candidate for a turnpike. Thus, instead of a freeway or turnpike, the relevant metaphor here is that of an air- or sea-lane through which journeys are routed, even though those lanes may not be the most direct route. Within the lane there are many possible routes, and which particular route is taken on one occasion is not the most relevant consideration for another. From a technical point of view, we then substitute a set for a point to obtain a non-trivial generalisation of the theory that reduces to the standard one when the set shrinks to a point and the sufficient conditions of the result are automatically activated.

For a field of inquiry to live, it must furnish problems, substantive and technical, to be worked on, and must have a space for its advances to be categorised and catalogued. The clearing of terminological confusion is important only in so far as it advances this end. Whether an economy, perfectly or imperfectly competitive, tends in the long run to a stationary state, is a time-honoured and classical question that surely dates to the 19^{th} century, if not earlier. The turnpike conjecture of the mid- 20^{th} century is different. It asks whether an allocation of resources, efficient from the viewpoint of one generation, however measured, must be near a stationary state, or some other benchmark which is generation-independent, given what each generation has received, and given what it has perforce to leave. Put another way, and again leaving aside a precise definition of how long a generation is, its theorems answer how generational well-being is to be ensured under the constraint that one bequeath to the future the world no worse than the one received. It is now recognised that our own century is beset with a whole host of multi-faceted environmental difficulties, and given that population and institutional design are once again being conceived as manipulable instruments of policy, turnpike theory surely has a bright future in it; see S220–S222, S234–S236, K6–K7, K9–K14 in Volume 2 and the relevant essays in (Aghion and Durlauf, 2005) for hints. In any case, in an age of computation and experimental mathematics (Barnsley, 2006, Borwein and Bailey, 2008), asymptotic convergence, and the *long run* that it implies, needs approximation to be rendered operational and policy-relevant, thereby guaranteeing an inevitable slide of the first question into the second, even for conventional fields of inquiry as in Yano (Yano, 1990, Yano, 1998) or in McKenzie's (McKenzie, 1998) singling out of new growth theory. In conclusion then, what began as a 25-year re-reading and updating of a 1987 entry, has, by necessity, had to read a narrative now over 60 years old, a reading that brings out its substantive and technical relevance involving computation for a variety of issues; capital-theoretic to be sure, but broadened enough to pertain to both intertemporal allocation and preservation of resources, including climate.

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See also

Ramsey model;
turnpike theory;
von Neumann ray

References

Aghion, P. and Durlauf, S. N. (eds.) 2005. *Handbook of Economic Growth*, Vols. 1 and 2. North-Holland, Amsterdam.
Arkin, V. and Evstigneev, I. 1987. *Stochastic Models of Control and Economic Dynamics*. Academic Press, New York.
Arrow, K. J. and Hahn, F. H. 1971. *General Competitive Analysis*. Holden Day, San Francisco.
Arrow, K. J., Hurwicz, L. and Uzawa, H. (eds.) 1958. *Studies in Linear and Non-Linear Programming*. Stanford University Press, Palo Alto.
Barnsley, M. F. 2006. *Superfractals*. Cambridge University Press, Cambridge.
Benhabib, J. 1992. *Cycles and Chaos in Economic Equilibrium*. Princeton University Press, Princeton.
Bewley, T. F. 2007. *General Equilibrium, Overlapping Generations Models and Optimal Growth Theory*. Harvard University Press, Cambridge.
Boldrin, M. and Montrucchio, L. 1986. Cyclic and chaotic behavior in intertemporal optimization models. *Mathematical Modelling*, 8, 697–700.
Boldrin, M. and Montrucchio, L. 1986. On the indeterminacy of capital accumulation paths. *Journal of Economic Theory*, 40, 26–39.
Bhattacharya, R. and Majumdar, M. 2004. Dynamical systems subject to random shocks: an introduction. *Economic Theory*, 23, 1–12.
Bhattacharya, R. and Majumdar, M. (eds.) 2007. *Random Dynamical Systems*. Cambridge University Press, Cambridge.
Borwein, J. and Bailey, D. 2008. *Mathematics by Experiment*, 2nd edn. A. K. Peters, Wellesley.
Carlson, D. A., Haurie, A. B. and Leizarowitz, A. 1991. *Infinite Horizon Optimal Control: Deterministic and Stochastic Systems*. Springer-Verlag, Berlin.
Dana, R. A., Le Van, C., Mitra, T. and Nishimura, K. (eds.) 2006. *Handbook of Optimal Growth*, Vol. 1. Springer, Berlin.
Debreu, G. 1983. *Mathematical Economics*. Cambridge University Press, Cambridge.
Deneckere, R. and Pelikan, S. 1986. Competitive chaos. *Journal of Economic Theory*, 40, 13–25.

Fujio, M. 2009. Optimal transition dynamics in the Leontief two-sector growth model with durable capital: the case of capital intensive consumption goods. *Japanese Economic Review*, 60, 490–511.

Dorfman, R., Solow, R. M. and Samuelson, P. A. 1958. *Linear Programming and Economic Analysis*. McGraw-Hill Book Co., New York.

Follmer, H. and Majumdar, M. 1978. On the asymptotic behavior of stochastic economic processes. *Journal of Mathematical Economics*, 5, 275–87.

Gale, D. 1970. Nonlinear duality and qualitative properties of optimal growth. In: *Integer and Nonlinear Programming* (ed. J. Abadie), pp. 309–19. North-Holland, Amsterdam.

Joshi, S. 1998. Turnpike theorems in nonconvex nonstationary environments. *International Economic Review*, 28, 225–48.

Karlin, S. J. 1960. *Mathematical Methods and Theory in Games, Programming and Economics*. Addison-Wesley, Reading MA.

Khan, M. Ali. 2005. Intertemporal ethics, modern capital theory and the economics of forestry. In: *Sustainability, Economics and Natural Resources: Economics of Sustainable Forest Management* (eds. S. Kant and A. Berry), Chapter 2, pp. 39–65. Springer, Netherlands.

Khan, M. Ali. 2010. La concorrenza perfetta come teoria dell'equilibrio. In: *La Matematica* (eds. C. Bartocci and P. Odifreddi), Vol. IV. Einaudi, Rome. (English translation available as *Perfect Competition as Equilibrium Theory*.)

Khan, M. Ali and Mitra, T. 2005. On choice of technique in the Robinson–Solow–Srinivasan model. *International Journal of Economic Theory*, 1, 83–109.

Khan, M. Ali and Mitra, T. 2010. Discounted Optimal Growth in the Two-Sector RSS Model: a Further Geometric Investigation. Johns Hopkins University, mimeo.

Khan, M. Ali and Mitra, T. 2011. Complicated Dynamics and Parametric Restrictions in the Robinson-Solow-Srinivasan Model. Cornell University, mimeo.

Khan, M. Ali and Mitra, T. 2012. Long run optimal behavior in a two-sector Robinson–Solow–Srinivasan model. *Macroeconomic Dynamics*, 16, 70–85.

Khan, M. Ali and Mitra, T. 2012. Impatience and dynamic optimal behavior: a bifurcation analysis of the Robinson–Solow–Srinivasan model. *Nonlinear Analysis*, 75, 1400–18.

Khan, M. Ali and Piazza, A. 2010. On uniform convergence of undiscounted optimal programs in the Mitra–Wan forestry model: the strictly concave case. *International Journal of Economic Theory*, 6, 57–76.

Khan, M. Ali and Piazza, A. 2010. On the non-existence of optimal programs in the Robinson–Solow–Srinivasan (RSS) model. *Economics Letters*, 109, 94–8.

Khan, M. Ali and Piazza, A. 2011. Classical Turnpike Theory and the economics of forestry. *Journal of Behavioral Economics and Organization*, 79, 194–201.

Khan, M. Ali and Piazza, A. 2011. The concavity assumption on felicities and asymptotic dynamics in the RSS model. *Set-Valued and Variational Analysis*, 19, 135–56.

Khan, M. Ali and Piazza, A. 2011. Optimal cyclicity and chaos in the 2-sector RSS model: an anything-goes construction. *Journal of Economic Behavior and Organization*, 80, 397–417.

Khan, M. Ali and Piazza, A. 2011. An overview of turnpike theory: towards the discounted deterministic case. *Advances in Mathematical Economics*, 14, 39–67.

Khan, M. Ali and Piazza, A. 2011. The economics of forestry and a set-valued turnpike of the classical type. *Nonlinear Analysis*, 74, 171–81.

Khan, M. Ali and Piazza A. 2012. On the Mitra–Wan forestry model: a unified analysis. *Journal of Economic Theory*, 147, 230–60.

Khan, M. Ali and Zaslavski, A. J. 2006. On a uniform turnpike of the third kind in the Robinson–Solow–Srinivasan model. *Journal of Economics*, 92, 137–66.

Khan, M. Ali and Zaslavski, A. J. 2009. On existence of weakly maximal programs: the RSS model with non-concave felicities. *Journal of Mathematical Economics*, 45, 624–33.

Khan, M. Ali and Zaslavski, A. J. 2010. On two classical turnpike results for the Robinson–Solow–Srinivasan (RSS) model. *Advances in Mathematical Economics*, 13, 47–97.

Koopmans, T. C. 1970. *Scientific Papers of Tjalling C. Koopmans*, Vols. 1 and 2. Springer-Verlag, Berlin.

Majumdar, M. and Nermuth, M. 1982. Dynamic optimization in non-convex models with irreversible investment: monotonicity and turnpike results. *Zeitschrift für Nationalökonomie*, 42, 339–62.

Majumdar, M. Mitra, T. and Nishimura, K. (eds.) 2000. *Optimization and Chaos*. Springer-Verlag, Berlin.
May, R. B. 1976. Simple mathematical models with very complicated dynamics. *Nature*, 40, 459–67.
McKenzie, L. W. 1986. Optimal economic growth, turnpike theorems and comparative dynamics. In: *Handbook of Mathematical Economics*, Vol. 3. (eds. K. J. Arrow and M. Intrilligator), pp. 1281-1355. North-Holland, New York.
McKenzie, L. W. 1987. Turnpike theory. In: *The New Palgrave* (eds. J. Eatwell, M. Milgate and P. K. Newman). MacMillan, London.
McKenzie, L. W. 1998. Turnpikes. *American Economic Review, Papers and Proceedings*, 88, 1–14.
McKenzie, L. W. 2002. *Classical General Equilibrium Theory*. The MIT Press, Cambridge.
Mitra, T. 2005. Characterization of the turnpike property of optimal paths in the aggregative model of intertemporal allocation. *International Journal of Economic Theory*, 1, 247–75.
Samuelson, P. A. 1965–2011. *The Collected Scientific Papers of Paul A. Samuelson*, Vols. 1–7. MIT Press, Cambridge.
Simons, S. 2008. *From Hahn– Banach to Monotonicity*. Springer-Verlag, Berlin.
Yano, M. 1990. von Neumann facets and the dynamic stability of perfect foresight equilibrium paths in neo-classical trade models. *Journal of Economics*, 51, 27–96.
Yano, M. 1998. On the dual stability of a von Neumann facet and the inefficacy of temporary fiscal policy. *Econometrica*, 66, 427–51.
Zaslavski, A. J. 2005. *Turnpike Properties in the Calculus of Variations and Optimal Control*. Springer, New York.
Zaslavski, A. J. 2009. Two turnpike results for discrete-time optimal control systems. *Nonlinear Analysis*, 71, 902–9.
Zaslavski, A. J. 2010. Structure of approximate solutions for discrete-time optimal control systems arising in economic dynamics. *Nonlinear Analysis*, 73, 952–70.