The optimal harvesting problem with a land market: a characterization of the asymptotic convergence*

Adriana Piazza†

Abstract

We study the asymptotic behavior of the optimal harvesting policies for a multiple species forest with a land market, i.e., any fraction of the land can be traded at any time stage. We prove the existence of *sustainable states* and we discuss the conditions under which any optimal trajectory converges in the long run towards one of these states or towards an optimal periodic cycle. We also discuss briefly a more general problem that includes costs of converting land between the different species.

Key words. forest management, asymptotic convergence, Lyapunov stability, turnpike theorem, non-linear discrete time model, age classes.

JEL Classification Numbers. C62, D90, Q23.

In abstract terms our model concerns the optimal management of a finite resource which can be allocated to different activities that, after a fixed delay, provide a benefit and liberate the resource for immediate reuse. At any time stage, any fraction of the resource can be traded. We study in particular, the optimal harvesting problem where the resource is the land surface that may be allocated among several forest species.

In 1849, Martin Faustmann correctly specified the problem of finding the economic value of a unique species, even-aged forest stand [3], a question solved by Ohlin in 1921 proving the existence of an optimal rotation period and characterizing the so-called *Faustmann age* [12]. The simplicity of Ohlin's result stems from the fact that he dealt with forests of identically aged trees. The generalization of the optimal rotation problem to a forest with many even-aged stands was already considered at Faustmann's time, but its complete resolution remains open even today. Nevertheless, Faustmann's ideas were extremely influential and inspired various harvesting rules which present a long run behavior that guarantees a sustainable and regular flow of timber. Samuelson provides an analysis of several issues in forestry economics, that is contained in [17]. Today, the forest economics literature proposes many different ways of tackling the problem. Solutions range from simple heuristics, simulations, linear optimization models, optimal control problems, exhaustive search of solutions, etc. Most of these works assume *a priori* that the desirable long-run state of the forest's population is some even land allocation between tree stands or include different types of even-flow constraints.

In the remarkable work of Mitra and Wan [9, 10], the problem of a multi-aged single species forest is examined analytically without any assumptions about the asymptotic behavior of the forest population or the timber flow. They prove the existence of a state invariant under the optimal policy, the so-called *sustainable state* or *normal forest*, but they also show that the classical belief about the long-run behavior of the forest state is not true: in the general case there is no convergence towards a state where the land is evenly allocated between tree stands, neither there is a stabilization of the timber flow. If the benefit function is linear every optimal trajectory is periodic after the first step, both in the discounted and undiscounted case, hence no convergence can be expected. If the benefit function is strictly concave and utilities are undiscounted, there is always asymptotic convergence of the optimal trajectories towards the sustainable state. The asymptotic behavior of the optimal solution of the discounted harvesting problem with a concave

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†Centro de Modelamiento Matemático. Universidad de Chile. Blanco Encalada 2120, 7° Piso. Santiago, Chile. E-mail: apiazza@dim.uchile.cl

function is not completely characterized by Mitra and Wan, but they provide examples where the optimal solution is a periodic cycle. Hence, no global convergence to the sustainable state can be expected.

More recently, Salo and Tahvonen [14]-[16] and Tahvonen [18] improve the results concerning the discounted, strictly concave problem using mathematical programming techniques. Among other results, they recover the existence of the sustainable state and show that there exists a neighborhood of it (V) such that every initial state belonging to V, yields an optimal trajectory that is periodic after the first step. This implies that the sustainable state is not even a local attractor. They conjecture that any optimal trajectory reaches V after finitely many steps, becoming periodic afterwards. The conjecture is proved for a two stands forest.

The same authors extend the model to include the possibility of allocating some land to an annual alternative use, proving that the optimal periodic cycles disappear when it is optimal to allocate part of the land to the annual use. In this framework, they prove that the sustainable state is a local attractor. Given that the total occupied surface is constant, the area allocated to the alternative use can be expressed as a function of the area allocated to the forest species. This fact is key in the proofs, making very difficult to study the general case (with several species and/or arbitrary maturity ages) with the same type of arguments.

The optimal management of a one species forest is also studied by Rapaport, Sraidi and Terreaux [13]. They consider a single species forest where harvest is forbidden before the maturity age and such that the tree's value remains constant after this age. They define a *greedy policy* as one where each tree is harvested as soon as it reaches maturity, showing that every optimal trajectory becomes greedy and periodic after a finite time, which means that the problem can be reduced to a finite dimensional one and easily solved.

In [2], Cominetti and Piazza extend the model of Rapaport et al. to a mixed forest with k-species having different maturity ages. They prove the existence and uniqueness of a sustainable state and provide sufficient conditions under which this state is a global attractor for the optimal trajectories. If these conditions are not satisfied the sustainable state is still a fixed point but optimal trajectories converge to the larger set of optimal periodic cycles.

Asymptotic convergence of optimal trajectories, also known as *turnpike properties*, have been established for wide classes of dynamic optimization problems, most of them issued from the literature on economic growth models. For a survey of the general results available we refer to Le Van et al. [4], Mc Kenzie [6, 7] and Zaslavski [19] and references therein. A distinguishing feature in [2] is that global convergence is obtained in a discounted utility framework with no restriction on the discount factor, while most turnpike theorems are either local or assume discount factors close to one. This is a notable fact since dynamic optimization models under strong discounting often exhibit a complex behavior including chaos (see, for example, Boldrin [1], Le Van et al. [4], Majumdar et al. [5], Mitra and Nishimura [8] and Montrucchio [11]). A factor that explains this more regular behavior is that the forest evolution has a natural periodic structure determined by the least common multiple N of the maturity ages of the species involved.

In the present paper we extend the model in [2] to include the possibility of buying and selling land at any time step. This maybe a useful tool to evaluate whether the surface occupied by a forest population is adequate. The analysis becomes more difficult because the uniqueness of the sustainable state is lost and new optimal periodic cycles appear, as we show in §2. Nevertheless, we prove that the main result is still valid under very general conditions on the maturity ages: optimal trajectories converge towards one of the sustainable states or to the set of optimal periodic cycles when the optimal trajectory becomes greedy (§ 3.2), when the land price is strictly concave (§ 3.3) and when this price is linear (§ 4). In every case, we show that land trading converges to zero and when the land price is linear this convergence occurs in finite time. Finally, in §5 we briefly discuss the asymptotic behavior of a forest where land conversion between species is costly. This problem is a generalization of the main topic of this article and the same type of asymptotic behavior is expected. This issue has been already considered by Salo and Tahvonen in [16] for the case of a one-species forest with an alternative annual use, presenting some numerical examples where conversion costs induce new optimal periodic cycles but with no analytical work.

1 Model formulation

We consider a discrete time model for the optimal management of a forest composed by k species $I = \{1, \ldots, k\}$ with maturity ages of $n_1, \ldots, n_k$ years respectively. For each period $t \in \mathbb{N}$ we denote $x_t^i \geq 0$ the area of species $i \in I$ reaching maturity in year t, while $\bar{x}_t^i \geq 0$ represents the area with trees beyond maturity (older than n_i). We must decide how much area $u_t^i \geq 0$ to harvest and how much land $c_t \in \mathbb{R}$ to trade on the market, after which the available land is allocated to new seedlings.

Let $S > 0$ denote the total land area and $a_t = \sum_{i \in I} \left[\bar{x}_t^i + \sum_{j=0}^{n_i - 1} x_{t+j}^i\right]$ the area actually occupied at time t. Taking $c_t > 0$ if some land is bought and $c_t < 0$ if it is sold, the occupied land evolves simply as

$$a_{t+1} = a_t + c_t, \qquad 0 \leq a_t \leq S. \tag{1}$$

Assuming that only mature trees can be harvested we must have $u_t^i \leq \bar{x}_t^i + x_t^i$. The area of mature trees not harvested in one period will comprise the trees beyond maturity at the following step

$$\bar{x}_{t+1}^i = \bar{x}_t^i + x_t^i - u_t^i. \tag{2}$$

The area made available after the harvest $\sum_{i \in I} u_t^i + c_t$ is allocated to new seedlings that will reach maturity in years $t + n_i$ respectively. This is expressed by the equation

$$\sum_{i \in I} x_{t+n_i}^i \quad = \quad c_t + \sum_{i \in I} u_t^i. \tag{3}$$

The benefit obtained from the harvest is $\sum_{t=0}^{\infty} \sum_{i \in I} b^t U_i(u_t^i) + b^t W(a_t, c_t)$ where $b \in (0, 1)$ is a discount rate and $U_i : \mathbb{R} \to \mathbb{R}$ are smooth, increasing and strictly concave functions that represent the benefit rendered by each of the forest species. The cost of land transactions is modelled as

$$W(a, c) = -\int_a^{a+c} \rho(\xi) d\xi - g(c)$$

where $\rho(\cdot) : [0, S] \to \mathbb{R}_+$ is a continuous, non-decreasing price function which takes into account that the scarcer the land is, the more expensive it becomes, while the term $-g(c)$ incorporates transaction costs such as administrative expenses. We take $g : \mathbb{R} \to \mathbb{R}$ convex, non-negative and such that $g(0) = 0$ [1]. Under these hypotheses, the following limits exist

$$\gamma_- = \lim_{\epsilon \to 0^-} \frac{g(\epsilon)}{\epsilon} \leq 0\,, \qquad \gamma_+ = \lim_{\epsilon \to 0^+} \frac{g(\epsilon)}{\epsilon} \geq 0$$

and $\partial g(0) = [\gamma_-, \gamma_+]$. We also require $\rho(0) + \gamma_- > 0$, which assures that $W(\cdot, c) > 0$ whenever $c < 0$, i.e., positive benefit is always obtained when land is sold.

We denote $\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i, \boldsymbol{a}, \boldsymbol{c}$ the sequences of states and controls. Since all areas are smaller than S and only $\boldsymbol{c}$ can take negative values, being bounded below by $-S$, it follows that these sequences belong to ℓ^∞.

An alternative representation of the forest in terms of the age distribution at time t is provided by the *state* $\mathbb{X}_t = (X_t^1, \ldots, X_t^k)$ where $X_t^i = (x_{t+n_i-1}^i, x_{t+n_i-2}^i, \ldots, x_t^i, \bar{x}_t^i)$ describes the areas occupied in year t by trees of species i with ages $1, 2, \ldots, n_i$ and over n_i. The state evolution consists of an age-shift dynamics, except for the first and last components of each vector X_t^i which are controlled by the harvesting-sowing policy.

In summary, the decision variables at each stage are

- c_t the area of land to be traded,
- u_t^i the area of mature trees that is harvested, $u_t^i \leq \bar{x}_t^i + x_t^i$,
- $x_{t+n_i}^i$ the area of land allocated to new seedlings of species i, satisfying $\sum_{i \in I} x_{t+n_i}^i = \sum_{i \in I} u_t^i + c_t$,

[1] A simple example of such functions is $g(c) = \gamma|c|$

and a typical evolution of species i can be represented as

$$X_t^i = (x_{t+n_i-1}^i, x_{t+n_i-2}^i, \ldots, x_{t+1}^i, x_t^i, \bar{x}_t^i) \to X_{t+1}^i = (x_{t+n_i}^i, x_{t+n_i-1}^i, \ldots, x_{t+2}^i, x_{t+1}^i, \bar{x}_t^i + x_t^i - u_t^i).$$

Although we will not use these dynamics explicitly, the state $\mathbb{X}_t$ will be useful in describing the asymptotic behavior of the forest.

Definition 1 *We denote Δ the set of all the states $\mathbb{X}_t \in \mathbb{R}_+^{\sum_{i\in I}(n_i+1)}$ such that $\sum_{i\in I}[\bar{x}_t^i + \sum_{j=t}^{t+n_i-1} x_j^i] \leq S$. We also denote Δ^0 the set of states with $\bar{x}_0^i = 0$ for all $i \in I$.*

Clearly enough the constraints (1)-(3) imply that $\mathbb{X}_t \in \Delta$ for all $t \in \mathbb{N}$, provided that $\mathbb{X}_0 \in \Delta$. Notice that we do not control $\mathbb{X}_0$ which corresponds to the initial state reflecting the age-class composition of the forest at time $t = 0$. Evidently, every initial state $\mathbb{X}_0$ yields a different set of feasible sequences $\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i, \boldsymbol{a}$ and $\boldsymbol{c}$. Hence, the optimization problem to be solved is parameterized with respect to the initial state and it may be stated as

$$P(\mathbb{X}_0) \quad \begin{cases} \text{maximize} & \sum_{t=0}^{\infty} b^t \big[\sum_{i\in I} U_i(u_t^i) + W(a_t, c_t)\big] \\ \text{subject to (1)-(3)} & \\ \boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i, \boldsymbol{a} \in \ell_+^\infty,\ \boldsymbol{c} \in \ell^\infty \text{ with } \mathbb{X}_0 \text{ given.} & \end{cases}$$

We observe that an initial state $\mathbb{X}_0 \in \Delta$ yields the same optimal value and harvesting policy as $\tilde{\mathbb{X}}_0 \in \Delta^0$ where $\tilde{X}_0^i = (x_{n_i-1}^i, x_{n_i-2}^i, \ldots, \bar{x}_0^i + x_0^i, 0)$.

Proposition 2 *For each $\mathbb{X}_0 \in \Delta$ the problem $P(\mathbb{X}_0)$ has an optimal solution.*

This proposition is a consequence of the Weierstrass theorem because the feasible set is non empty and $\sigma(\ell^\infty, \ell^1)$-compact[2] while the objective function is $\sigma(\ell^\infty, \ell^1)$-upper semi continuous. The proof is a straightforward adaptation of [2] where a simpler problem with no land trading is treated. We observe that the harvests $\boldsymbol{u}^i$ are unique due to the strict concavity of U_i. Hence, we can assure the uniqueness of the $\boldsymbol{x}^i$ when restricted to so-called greedy trajectories where $\bar{\boldsymbol{x}}^i = 0$. Uniqueness of $\boldsymbol{a}$ and $\boldsymbol{c}$ follows as well from (1) and (3).

Before going on with the resolution of the problem, we express it in a different way. The objective function can be stated as

$$\begin{aligned} V &= \sum_t b^t \big[\sum_{i=1}^k U_i(u_t^i) - \int_0^{a_{t+1}} \rho(\xi)d\xi + \int_0^{a_t} \rho(\xi)d\xi - g(c_t)\big] \\ &= \tfrac{1}{b}\int_0^{a_0} \rho(\xi)d\xi + \sum_t b^t \big[\sum_{i=1}^k U_i(u_t^i) - \tfrac{1-b}{b}\int_0^{a_t} \rho(\xi)d\xi - g(c_t)\big] \end{aligned}$$

where the first term depends only on the initial condition. Also, defining the variable $x_t^0 = S - a_t$ which represents the unused area, we get $c_t = x_t^0 - x_{t+1}^0$ and the area balance can be written as

$$\sum_{i\in I} x_{t+n_i}^i + x_{t+1}^0 = x_t^0 + \sum_{i\in I} u_t^i.$$

This suggests to consider the unused land as a new species X^0 with benefit function

$$U_0(x^0) = -\tfrac{1-b}{b}\int_0^{S-x^0} \rho(\xi)d\xi$$

and maturity age $n_0 = 1$. Notice that $U_0(\cdot)$ is smooth, increasing and concave as a function of x^0. The fact that U_0 is negative is of no importance to our proofs. With these definitions we may restate the problem as

$$P(\mathbb{X}_0) \quad \begin{cases} \text{maximize} & \sum_{t=0}^{\infty} b^t \big[\sum_{i=0}^k U_i(u_t^i) - g(c_t)\big] \\ \text{subject to} & \bar{x}_{t+1}^i = \bar{x}_t^i + x_t^i - u_t^i \quad i \in \{0, \cdots, k\} \\ & \sum_{i=0}^k x_{t+n_i}^i = \sum_{i=0}^k u_t^i \\ & x_{t+1}^0 = x_t^0 - c_t \\ & \boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i \in \ell_+^\infty, \boldsymbol{c} \in \ell^\infty \text{ with } \mathbb{X}_0 \text{ given.} \end{cases}$$

Notice that having $n_0 = 1$ and U_0 increasing assures that $\bar{\boldsymbol{x}}^0 = 0$ and $\boldsymbol{u}^0 = \boldsymbol{x}^0$ at the optimum.

[2] The feasible set is a closed convex subset of the ℓ^∞ ball of radius S centered at the origin.

2 Stationary optimal trajectories

In this section we characterize the initial states $\mathbb{X}_0$ that give rise to stationary or periodic optimal trajectories with no land trading. In the next section we discuss conditions under which every optimal trajectory converges to this particular set of states.

2.1 Sustainable states

We begin by introducing the notion of a sustainable state, which corresponds intuitively to a forest with an age distribution at which it is optimal to stay forever, i.e., the maximum benefit for $P(\mathbb{X}_0)$ can be obtained with a harvesting-sowing policy under which the state is $\mathbb{X}_t = \mathbb{X}_0$ for all t.

Definition 3 *A state $\mathbb{X} \in \Delta$ is called* sustainable *if it is invariant under an optimal harvesting-sowing policy. The set of sustainable states is denoted Δ^*.*

The existence of sustainable states is not completely obvious. Clearly any such state must be of the form $X^i = (x^i, \dots, x^i, \bar{x}^i)$ with an invariant optimal harvesting-sowing policy: harvest x^i and sow exactly the same area in order to keep an invariant configuration. It is also clear that we must have $\bar{x}^i = 0$, that is to say $\mathbb{X} \in \Delta^0$, since otherwise a policy that harvests a little more at time $t = 0$ and x^i in all other periods would provide a greater benefit contradicting optimality. For the rest of this paper we denote $\sigma_i = b^{n_i}/(1-b^{n_i})$ and without loss of generality we assume that the species are ordered in such a way that $\sigma_1 U_1'(0) \geq \sigma_2 U_2'(0) \geq \ldots \geq \sigma_k U_k'(0)$.

In order to characterize the sustainable states, for each area $a \in [0, S]$ we denote $\mathbb{X}_a^* \in \Delta^0$ the homogeneous state $X_a^i = (x_a^{*i}, \dots, x_a^{*i}, 0)$ where x_a^* is the solution to the strictly concave problem

$$(S_a) \quad \begin{cases} \text{maximize} & \sum_{i\in I} n_i \sigma_i U_i(x^i) \\ \text{subject to} & x^i \geq 0 \text{ and } \sum_{i\in I} n_i x^i = a. \end{cases}$$

We denote $I_a^* = \{i \in I : x_a^{*i} > 0\}$ the species present in $\mathbb{X}_a^*$ and we let r_a be the Lagrange multiplier associated to the area constraint $\sum_{i\in I} n_i x_a^i = a$, so that the optimal solution is characterized by $\sigma_i U_i'(x_a^{*i}) = r_a$ for $i \in I_a^*$ and $\sigma_j U_j'(0) \leq r_a$ for $j \notin I_a^*$. The ordering of the species and the strict concavity of U_i then imply that $I_a^* = \{1, \dots, i_a^*\}$ for some index i_a^* and the equality $r_a = \sigma_1 U_1'(x_a^{*1})$. We remark that when a increases all the optimal areas x_a^{*i} increase while the multiplier r_a decreases strictly.

We will prove that every sustainable state is of the form $\mathbb{X}_a^*$ for some area $a \in [0, S]$. However not every surface a supports a sustainable state. Let us introduce the auxiliary function δ

Definition 4 *Let $\delta : [0, S] \to \mathbb{R}$ be the decreasing function $\delta(a) = r_a - \rho(a)$* [3] *and denote $[\underline{a}, \overline{a}]$ the interval which comprises all the areas $a \in (0, S)$ such that $\gamma_- \leq \delta(a) \leq \gamma_+$, together with 0 if $\delta(0) \leq \gamma_+$ and S if $\delta(S) \geq \gamma_-$. An alternative expression for $\delta(a)$ is $\sigma_1 U_1'(x_a^{*1}) - \sigma_0 U_0'(S - a)$.*

Notice that when $\delta(0) \leq \gamma_-$ we have a degenerate interval $[\underline{a}, \overline{a}] = \{0\}$, while when $\delta(S) \geq \gamma_+$ we get $[\underline{a}, \overline{a}] = \{S\}$. In all other cases $[\underline{a}, \overline{a}]$ is always non-empty and non-degenerate.

Proposition 5 $\Delta^* = \{\mathbb{X}_a^* : a \in [\underline{a}, \overline{a}]\}$.

Proof. Let us prove that $\mathbb{X}_a^*$ is sustainable for all $a \in [\underline{a}, \overline{a}]$. Take $a \in (0, S)$ with $\gamma_- \leq \delta(a) \leq \gamma_+$ and consider the stationary trajectory issued from the initial condition $\mathbb{X}_0 = \mathbb{X}_a^*$, i.e.,

$$x_t^i = u_t^i = x_a^{*i} \text{ for all } t, \ \bar{\boldsymbol{x}}^i = 0 \text{ and } \boldsymbol{c} = 0. \tag{4}$$

In order to prove its optimality for $P(\mathbb{X}_a^*)$ we introduce the Lagrangian

[3] We recall that $\rho(a)$ was assumed to be a non-decreasing function. From this and the fact that r_a is decreasing, we get directly that $\delta(a)$ is decreasing.

$$\begin{aligned} L &= \textstyle\sum_{t=0}^{\infty} \{ b^t \big[\sum_{i=0}^{k} U_i(u_t^i) - g(c_t) \big] + \sum_{i=0}^{k} \mu_t^i u_t^i \} + \sum_{i=0}^{k} \big[\sum_{t=1}^{\infty} \bar{\lambda}_t^i \bar{x}_t^i + \sum_{t=n_i}^{\infty} \lambda_t^i x_t^i \big] \\ &\quad + \textstyle\sum_{t=0}^{\infty} \{ \sum_{i=0}^{k} \alpha_t^i (\bar{x}_t^i + x_t^i - u_t^i - \bar{x}_{t+1}^i) + \theta_t \sum_{i=0}^{k} (u_t^i - x_{t+n_i}^i) + \nu_t (x_t^0 - c_t - x_{t+1}^0) \} \end{aligned} \tag{5}$$

together with the following set of ℓ^1-multipliers

$$\begin{cases} \mu_t^i = \lambda_t^0 = 0 & \\ \theta_t = b^t r_a & \\ \alpha_t^i = \theta_t + b^t U_i'(x_a^{*i}) & i \in \{0, \cdots, k\} \\ \bar{\lambda}_t^i = \alpha_t^i (1-b)/b & i \in \{0, \cdots, k\} \\ \lambda_t^i = \frac{\theta_t}{\sigma_i} - b^t U_i'(x_a^{*i}) & i \in I \\ \nu_t = -b^t \delta(a). & \end{cases} \tag{6}$$

Observe that L is concave with respect to the primal variables $(\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i, \boldsymbol{c})$. To verify the stationarity of L, we must show that $0 \in \partial^+ L$, the upper subdifferential of L, at the point given by (4) while the multipliers are fixed at the values stated in (6). In fact, L is differentiable with respect to every variable except $\boldsymbol{c}$, and a long but straightforward computation shows that the partial derivatives $L_{u_t^i} = L_{x_t^i} = L_{\bar{x}_t^i} = 0$ for all i and t.[4] In addition, it is easy to see that

$$0 \in \partial_{c_t}^+ L = -b^t \partial g(0) - \nu_t \iff -\nu_t \in b^t \partial g(0) \iff \delta(a) \in [\gamma_-, \gamma_+]. \tag{7}$$

We can see directly that the complementary slackness is satisfied and that $\theta, \alpha^i, \bar{\lambda}^i, \lambda^i \in \ell_+^1$. Thus the proposed trajectory is a stationary point of the Lagrangian, hence a solution to $P(\mathbb{X}_a^*)$ and therefore $\mathbb{X}_a^*$ is sustainable.

Similarly, to show that $\mathbb{X}_0^*$ is a sustainable state when $\delta(0) \leq \gamma_+$, we take the multipliers

$$\begin{cases} \mu_t^i = \lambda_t^0 = 0 & \\ \theta_t = b^t [\sigma_0 U_0'(S) + \gamma_+] & \\ \alpha_t^i = \theta_t + b^t U_i'(x_0^{*i}) & i \in \{0, \cdots, k\} \\ \bar{\lambda}_t^i = \alpha_t^i (1-b)/b & i \in \{0, \cdots, k\} \\ \lambda_t^i = \frac{\theta_t}{\sigma_i} - b^t U_i'(0) & i \in I \\ \nu_t = -b^t \gamma_+. & \end{cases}$$

The stationarity of the Lagrangian is verified as before and the complementary slackness is straightforward, while the non-negativity of λ_t^i follows from $\lambda_t^i \geq \frac{b^t}{\sigma_i}[\gamma_+ - \delta(0)] \geq 0$. Finally, to see that $\mathbb{X}_S^*$ is a sustainable state when $\delta(S) \geq \gamma_-$, consider again the multipliers of (6) except for the values of λ_t^0 and ν_t

$$\begin{cases} \lambda_t^0 = \frac{b^t}{\sigma_0}[\delta(S) - \gamma_-] \geq 0 \\ \nu_t = -b^t \gamma_- . \end{cases}$$

Conversely, we show next that every sustainable state is of the form $\mathbb{X}_a^*$ for some $a \in [\underline{a}, \overline{a}]$. Let $\mathbb{X}$ be sustainable with $X^i = (x^i, \ldots, x^i, 0)$ and set $a = \sum_{i \in I} n_i x^i$. We claim that $x^i > 0$ implies $\sigma_i U_i'(x^i) \geq \sigma_j U_j'(x^j)$ for all $j \in I$. Indeed, let us perturb the optimal harvesting policy as follows: at time $t = 0$ we sow $x^i - \epsilon$ and $x^j + \epsilon$ instead of x^i and x^j, while in all subsequent periods we harvest all mature trees and sow the harvested areas with the same species they had. The benefit derived from this perturbed policy must be less than the one obtained with the optimal policy, which gives

$$\tfrac{b^{n_i}}{1-b^{n_i}}[U_i(x^i - \epsilon) - U_i(x^i)] + \tfrac{b^{n_j}}{1-b^{n_j}}[U_j(x^j + \epsilon) - U_j(x^j)] \leq 0.$$

[4] All the partial derivatives of L are evaluated at the point previously stated, i.e., the one defined by (4) and (6). For the sake of simplicity, we do not write the evaluation point every time.

Dividing by ϵ and letting $\epsilon \to 0$ we deduce $\sigma_i U_i'(x^i) \geq \sigma_j U_j'(x^j)$ as claimed. Using this fact and setting $I^* = \{i : x^i > 0\}$ it follows that $\sigma_i U_i'(x^i)$ is constant for $i \in I^*$ and larger than or equal to the value of this expression for $i \notin I^*$. This implies that the vector $(x^i)_{i\in I}$ is an optimal solution for (S_a) so that $x^i = x_a^{*i}$ and therefore $\mathbb{X} = \mathbb{X}_a^*$.

It is still left to see that $a \in [\underline{a}, \overline{a}]$, for which we must prove that $\delta(a) \leq \gamma_+$ if $a < S$ as well as $\delta(a) \geq \gamma_-$ if $a > 0$. Suppose first that $a < S$ and consider the following perturbation to the optimal stationary trajectory issued from $\mathbb{X}_a^*$: at time $t = 0$ buy an ϵ of land and sow $x_a^{1*} + \epsilon$ instead of x_a^{1*}. After that we continue with a periodic policy, harvesting all mature trees and sowing the liberated areas with the same species they had. The benefit obtained with the alternative trajectory must be less than or equal to the optimal one, which yields

$$\tfrac{b}{1-b}[U_0(x_a^{0*} - \epsilon) - U_0(x_a^{0*})] - g(\epsilon) + \tfrac{b^{n_1}}{1-b^{n_1}}[U_1(x_a^{1*} + \epsilon) - U_1(x_a^{1*})] \leq 0.$$

Dividing by ϵ and letting $\epsilon \to 0$ we deduce $-\sigma_0 U_0'(x_a^{0*}) - \gamma_+ + \sigma_1 U_1'(x_a^{1*}) \leq 0$, or equivalently $\delta(a) \leq \gamma_+$. Similarly, if $a > 0$ we consider the selling of an ϵ of land, i.e., we take $\epsilon < 0$ and repeat the same reasoning to obtain $\sigma_0 U_0'(x_a^{0*}) + \gamma_- - \sigma_1 U_1'(x_a^{1*}) \leq 0$ which is equivalent to $\delta(a) \geq \gamma_-$. ∎

2.2 Greedy Periodic Cycles

A trajectory is called *greedy* if all the mature trees are harvested at every stage. Such trajectories were introduced in [13] when studying a single species forest, in which case they yield periodic trajectories. This is no longer the case when multiple species are involved, since the sowing policy is not determined. This distinction was already made in [2], introducing a special class of greedy trajectories called *greedy periodic cycles*, in which all harvested areas are sown with the same species they had before the harvest. In the current setting, for the artificial species X^0 that represents the unused land, the latter condition imposes that no land is traded so that $c_t = 0$ and $x_t^0 = x^0$ for all t.

Definition 6 *A feasible trajectory* $(\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i)_{i\in I}$ *is called* greedy *if* $\bar{x}_t^i = 0$*. It will be called a* greedy periodic cycle (GPC) *if in addition* $x_{t+n_i}^i = u_t^i$*. We denote* Δ^g *the set of initial states* $\mathbb{X}_0 \in \Delta^0$ *for which there exists an optimal trajectory which is greedy, and* Δ^p *those having an optimal trajectory that is a* GPC.

It is worth mentioning that since the optimal harvests are uniquely determined, the area balance implies that an optimal trajectory issued from an initial state $\mathbb{X}_0 \in \Delta^g$ must be unique, namely a greedy one, and therefore Δ^g is forward invariant through optimal policies: $\mathbb{X}_t \in \Delta^g \Rightarrow \mathbb{X}_{t+1} \in \Delta^g$.

Theorem 7 *Let* $\mathbb{X}_0 \in \Delta^0$ *such that* $x^0 \in (0, S)$ *and consider the periodic sequences* $(x_t^i)_{t\in\mathbb{N}}$ *built from* $\mathbb{X}_0$ *with* $x_{t+n_i}^i = x_t^i$*. Then* $\mathbb{X}_0 \in \Delta^p$ *iff for all* $i, j \in I$ *and* $t \in \mathbb{N}$ *we have*[5]

$$\begin{array}{lll}
(a) & U_i'(x_t^i) \geq b\, U_i'(x_{t+1}^i) & \\
(b) & \sigma_i U_i'(x_t^i) \geq b^{n_j}\left[\sigma_i U_i'(x_{t+n_j}^i) + U_j'(x_t^j)\right] & \forall\, x_t^i > 0 \\
(c) & \sigma_1 U_1'(x_t^1) - \sigma_0 U_0'(x^0) \geq \gamma_- & \forall\, x_t^1 > 0 \\
(d) & \sigma_1 U_1'(x_t^1) - \sigma_0 U_0'(x^0) \leq \gamma_+ & \forall\, x_t^1
\end{array} \tag{8}$$

Proof. The arguments are similar to those used in [2, Theorem 3.4]. In particular we observe that conditions (a) and (b) are exactly the necessary and sufficient conditions that characterize Δ^p in [2], where the occupied surface is constant and given. The proof is based on [2, Lemma 3.5] which states that condition (b) implies

$$x_t^i > 0 \quad \Rightarrow \quad \sigma_i U_i'(x_t^i) \geq \sigma_j U_j'(x_t^j) \quad \textit{for } i, j \in I \tag{9}$$

[5]Notice that it suffices to check condition $a)$ for $t = 0, 1, \ldots, n_i - 1$. Similarly $c)$ and $d)$ must only be checked for $t = 0, 1, \ldots, n_1 - 1$, while for condition $b)$ it suffices to check it for $t = 0, 1, \ldots, n_{ij} - 1$ where n_{ij} is the least common multiple of the maturity ages n_i and n_j.

which in turn gives

$$x_t^i > 0 \Rightarrow x_t^j > 0 \text{ for all } j < i,\ i, j \in I \tag{10}$$

since otherwise (9) would yield the contradiction $\sigma_i U_i'(x_t^i) \geq \sigma_j U_j'(0) \geq \sigma_i U_i'(0) > \sigma_i U_i'(x_t^i)$. Combining (9) and (10) we get

$$x_t^i > 0 \Rightarrow \sigma_i U_i'(x_t^i) = \sigma_j U_j'(x_t^j) \text{ for all } j < i. \tag{11}$$

<u>SUFFICIENT CONDITION:</u> To establish the sufficiency take $\mathbb{X}_0$ satisfying (8) and consider the corresponding GPC. In order to prove its optimality if suffices to check that it is a stationary point for the Lagrangian (5) with the following set of multipliers

$$\begin{cases} \mu_t^i = \lambda_t^0 = 0 & \\ \theta_t = b^t \sigma_1 U_1'(x_t^1) & \\ \alpha_t^i = \theta_t + b^t U_i'(x_t^i) & i \in \{0, \cdots, k\} \\ \bar{\lambda}_t^i = \alpha_{t-1}^i - \alpha_t^i & i \in \{0, \cdots, k\} \\ \lambda_t^i = b^t \{\sigma_1 [\frac{1}{b^{n_i}} U_1'(x_{t-n_i}^1) - U_1'(x_t^1)] - U_i'(x_t^i)\}, & i \in I \\ \nu_t = b^t [\sigma_0 U_0'(x^0) - \sigma_1 U_1'(x_t^1)]. & \end{cases} \tag{12}$$

All these multipliers are of the form b^t multiplied by some bounded sequence so they belong to ℓ^1. The non-negativity of θ_t and α_t^i is evident and that of $\bar{\lambda}_t^i$ follows directly from (8)a. For $\lambda_t^i \geq 0$ we observe that this is assured by condition (8)b whenever $x_{t-n_i}^1 > 0$, while when $x_{t-n_i}^1 = 0$ using the monotonicity of U_i' and the fact that $\sigma_1 U_1'(0) \geq \sigma_i U_i'(0)$ we get

$$\frac{\lambda_t^i}{b^t} = \sigma_1 [\tfrac{1}{b^{n_i}} U_1'(0) - U_1'(x_t^1)] - U_i'(x_t^i) \geq \sigma_1 [\tfrac{1}{b^{n_i}} - 1] U_1'(0) - U_i'(x_t^i) \geq U_i'(0) - U_i'(x_t^i) \geq 0.$$

Therefore all the multipliers, except ν_t, belong to ℓ_+^1.

The complementary slackness is obvious except for the constraint $x_t^i \geq 0$ which follows from (11) because $x_t^i > 0$ implies $\sigma_i U_i'(x_t^i) = \sigma_1 U_1'(x_t^1) = \sigma_1 U_1'(x_{t-n_i}^1)$ and then $\lambda_t^i = 0$. Verification of stationarity is straightforward except for condition $0 \in \partial_{c_t}^+ L$ which is equivalent to $\nu_t \in b^t[\gamma_-, \gamma_+]$. This last condition is obviously assured by (c) and (d) when $x_t^1 > 0$. And of course, $\sigma_1 U_1'(0) - \sigma_0 U_0'(x^0) \geq \sigma_1 U_1'(x_t^1) - \sigma_0 U_0'(x^0) \geq \gamma_-$ which proves (c) for all t as long as $X^1 \neq 0$. But having $X^1 = 0$ is impossible since (10) would imply $X^i = 0$ for all $i \in I$ and $x^0 = S$. This completes the proof of optimality of the GPC for $P(\mathbb{X}_0)$.

<u>NECESSARY CONDITION:</u> To prove the necessity of condition (8) take $\mathbb{X}_0 \in \Delta^p$ so that the periodic sequences $(x_t^i)_{t \in \mathbb{N}}$ are optimal for $P(\mathbb{X}_0)$. The key idea is to compare the optimal benefit with those of a set of carefully chosen alternative trajectories.

We skip the proof of the necessity of (a) and (b) as it is identical to that of [2, Theorem 3.4].

To prove (8)(d) consider the following perturbation to the optimal trajectory: buy a fraction ϵ of land at time t and allocate it to the first species. The difference of benefit is

$$b^t \tfrac{b^{n_1}}{1-b^{n_1}} [U_1(x_t^1 + \epsilon) - U_1(x_t^1)] + b^t \tfrac{b}{1-b} [U_0(x^0 - \epsilon) - U_0(x^0)] - b^t g(\epsilon) \leq 0$$

Thus, dividing by $b^t \epsilon$ and making $\epsilon \to 0$ we obtain $\sigma_1 U_1'(x_t^1) - \sigma_0 U_0'(x^0) \leq \gamma_+$.

A similar argument yields (8)(c): if $x_t^1 > 0$, we sell an ϵ of the land allocated to the first species and we repeat the same procedure to get $\gamma_- \leq \sigma_1 U_1'(x_t^1) - \sigma_0 U_0'(x^0)$. ■

The following corollary is a direct consequence of (11).

Corollary 8 *Let* $I(\mathbb{X}_0) = \{i : X^i \neq 0\}$. *Then for all* $\mathbb{X}_0 \in \Delta^p$ *we have* $I(\mathbb{X}_0) = \{1, \ldots, i_0\}$ *for some* i_0.

It may be surprising that (c) and (d) only involve species 0 and 1. In fact, there is an equivalent symmetric characterization,

Proposition 9 *Conditions* (8) *(c) and (d) of the theorem above can be substituted by*

$$\begin{array}{ll} (c') & \sigma_i U_i'(x_t^i) - \sigma_0 U_0'(x^0) \geq \gamma_- \quad \forall\, x_t^i > 0,\; i \in I \\ (d') & \sigma_i U_i'(x_t^i) - \sigma_0 U_0'(x^0) \leq \gamma_+ \quad \forall\, x_t^i,\; i \in I \end{array} \tag{13}$$

Proof. From (11) it is clear that (8)(c) and (13)(c') are equivalent. For (d) we observe that either $x_t^1 > 0$ and $\sigma_1 U_1'(x_t^1) \geq \sigma_i U_i'(x_t^i)$ or $x_t^1 = x_t^i = 0$ and $\sigma_1 U_1'(0) \geq \sigma_i U_i'(0)$. In both cases, (8)($d$) implies (13)($d'$) while the implication in the other sense is evident. ■

Let us consider next the limiting cases $x^0 = 0$ and $x^0 = S$. When $x^0 = S$ all the state variables are zero and the corresponding greedy periodic cycle is the stationary trajectory starting from $\mathbb{X}_0^* = 0$, which is optimal if and only if $\delta(0) \leq \gamma_+$. For the case $x^0 = 0$ things are more involved. In fact, it is easy to see that conditions (8)(a), (b) and (c) are still necessary, but it turns out that they are not sufficient. Even more, we present a set of more restrictive necessary conditions which are not yet sufficient and a set of sufficient conditions. Both sets comprise conditions (8)(a) and (b) together with an ad hoc modification of (8)(c), while (d) is no longer considered as it is related to the possibility of buying land which is obviously not possible when $x^0 = 0$.

Theorem 10 *Consider a GPC issued from an initial state $\mathbb{X}_0 \in \Delta^0$ with $x^0 = 0$. The following conditions are sufficient to guarantee that $\mathbb{X}_0 \in \Delta^p$: for all $i, j \in I$ and $t \in \mathbb{N}$ we have*

$$\begin{array}{lll} (a) & U_i'(x_t^i) \geq b\, U_i'(x_{t+1}^i) & \\ (b) & \sigma_i U_i'(x_t^i) \geq b^{n_j} \left[\sigma_i U_i'(x_{t+n_j}^i) + U_j'(x_t^j)\right] & \forall\, x_t^i > 0 \\ (c) & \sigma_1 \left[U_1'(x_t^1) - b U_1'(x_{t+1}^1)\right] - b U_0'(x^0) - (1-b)\gamma_- \geq 0 & \forall\, x_t^1 > 0. \end{array} \tag{14}$$

A set of necessary conditions comprises (a), (b) *and the following inequality for all $i, j \in I$ and $k \in \mathbb{N}$* [6]

$$\sigma_i U_i'(x_t^i) - b^k \sigma_j U_j'(x_{t+k}^j) - (1-b^k)\sigma_0 U_0'(x^0) - \gamma_- + b^k \gamma_+ \geq 0 \quad \forall\, x_t^i > 0. \tag{15}$$

Proof. To prove the sufficiency of condition (14) it is enough to consider again the multipliers of (12) except for the values of λ_t^0 and ν_t

$$\left\{ \begin{array}{l} \lambda_t^0 = b^{t-1}\left\{\sigma_1\left[U_1'(x_{t-1}^1) - b U_1'(x_t^1)\right] - b U_0'(0) - (1-b)\gamma_-\right\} \\ \nu_t = -b^t \gamma_- \end{array} \right.$$

The concavity of U_1 implies that if (14)(c) holds at time t then it also holds at $t+1$ even if $x_{t+1} = 0$. As (10) forces $X^1 \neq 0$, we deduce that (14)(c) holds for all t and the non-negativity of λ_t^0 follows directly.

On the other hand, the necessity of (15) is easily seen by taking a perturbed trajectory that consists in selling ϵ of the land assigned to species i at time t and buying it back at time $t+k$ to assign it to species j, continuing afterwards with a GPC. The difference of benefit is

$$\begin{array}{l} b^t \frac{b^{n_i}}{1-b^{n_i}} [U_i(x_t^i - \epsilon) - U_i(x_t^i)] + b^{t+k} \frac{b^{n_j}}{1-b^{n_j}} [U_j(x_{t+k}^j + \epsilon) - U_j(x_{t+k}^j)] \\ + b^t (b + \ldots + b^k)[U_0(\epsilon) - U_0(0)] - b^t g(-\epsilon) - b^{t+k} g(\epsilon) \leq 0 \end{array}$$

Thus, dividing by $b^t \epsilon$ and letting $\epsilon \to 0$ we obtain

$$-\sigma_i U_i'(x_t^i) + b^k \sigma_j U_j'(x_{t+k}^j) + b \tfrac{1-b^k}{1-b} U_0'(0) + \gamma_- - b^k \gamma_+ \leq 0$$

which is exactly (15). ■

Remark 1 *Notice that* (15) *can be easily deduced from* (13)(c') *and* (d') *since*

$$\sigma_i U_i'(x_t^i) - \sigma_0 U_0'(x^0) - \gamma_- \geq 0 \geq b^k \left[\sigma_j U_j'(x_{t+k}^j) - \sigma_0 U_0'(x^0) - \gamma_+\right] \quad \forall x_t^i > 0 \;\Rightarrow\; (15).$$

This is the reason why it is not relevant when $x^0 \in (0, S)$.

[6] Notice that (8)(c) is retrieved by letting $k \to \infty$ in (15).

2.3 Relations between GPCs and sustainable states

Clearly, every stationary trajectory is a GPC so $\Delta^* \subseteq \Delta^p$. In the following we examine more closely the connections between greedy periodic cycles and sustainable states, pointing out a situation in which they coincide.

Proposition 11 *Let $\mathbb{X} \in \Delta^p$. Then $a = S - x^0 \in [\underline{a}, \overline{a}]$.*

Proof. By Definition 4 it suffices to check that $\delta(a) \geq \gamma_-$ if $a > 0$ and $\delta(a) \leq \gamma_+$ if $a < S$.

Suppose $a > 0$. If there is any $x_t^1 \geq x_a^{*1}$ then (8)(c) implies $\gamma_- \leq \sigma_1 U_1'(x_t^1) - \sigma_0 U_0'(x^0) \leq \sigma_1 U_1'(x_a^{*1}) - \sigma_0 U_0'(x^0) = \delta(a)$. Suppose by contradiction that $x_t^1 < x_a^{*1}$ for all t. Due to (11) and the characterization of the sustainable state we know that for all $x_t^i > 0$

$$\sigma_i U_i'(x_t^i) = \sigma_1 U_1'(x_t^1) > \sigma_1 U_1'(x_a^{*1}) \geq \sigma_i U_i'(x_a^{*i}) \Rightarrow x_t^i < x_a^{*i},$$

and thus the area balance cannot be satisfied, which is obviously absurd.

Suppose next $a < S$. If there is any $x_t^1 \leq x_a^{*1}$, then $\delta(a) \leq \gamma_+$ follows from (8)(d). Suppose that $x_t^1 > x_a^{*1}$ for all t, proceeding as before we conclude that for all $i \in I_a^*$ we have $\sigma_i U_i'(x_a^{*i}) = \sigma_1 U_1'(x_a^{*1}) > \sigma_1 U_1'(x_t^1) \geq \sigma_i U_i'(x_t^i)$, where the last inequality follows from (9). This gives $x_a^{*i} < x_t^i$ for all t and $i \in I_a^*$, and so the area balance cannot be satisfied and a contradiction arises. ■

As a direct consequence of this result and the commentary after Definition 4, it follows that when $\delta(0) \leq \gamma_-$ we have $\Delta^p = \Delta^* = \{\mathbb{X}_0^*\}$, while if $\delta(S) \geq \gamma_+$ then $\Delta^p \subseteq \{\mathbb{X} \in \Delta^0 : x^0 = 0\}$. The following result gives a sufficient condition to assure the equality. For the rest of the paper m.c.d. stands for *maximum common divisor*. From now on, we denote $\Delta^p(a) = \Delta^p \cap \{\mathbb{X} \in \Delta^0 : x^0 = S - a\}$.

Proposition 12 *If m.c.d.$\{n_i : i \in I_a\} = 1$ then $\Delta^p(a) = \{\mathbb{X}_a^*\}$. Thus, if m.c.d.$\{n_i : i \in I_{\underline{a}}\} = 1$ then $\Delta^p = \Delta^*$.*

Proof. Using [2, Theorem 3.8] we see that for a given surface a the first condition yields that $\mathbb{X}_a^*$ is the only state in Δ^p whose covered surface is equal to a. The second hypothesis implies m.c.d.$\{n_i : i \in I(\mathbb{X}_a^*)\} = 1$ for all $a \in [\underline{a}, \overline{a}]$, which readily implies the lemma. ■

3 Convergence of optimal trajectories

We turn next to the study of the long run behavior of the optimal harvesting policies. The previous section described some special states from which the optimal trajectory is either invariant or periodic. We conjecture that such behavior is typical in the sense that an optimally managed forest converges either to a sustainable state or to the set of optimal GPCs. To prove such a *global attractor property* we rely on a suitable Lyapunov function to analyze the asymptotic behavior of optimal trajectories that become greedy after a finite time. We then present some sufficient conditions under which this is effectively the case. More importantly, we prove that when U_0 is strictly concave this result holds not only for greedy trajectories but for every optimal harvesting policy. We also show that land trading converges to zero and the occupied surface converges to a limit. In the following section we treat the case of U_0 linear, retrieving stronger results: the convergence occurs after finitely many steps. In the very particular case of linear land price and a function g differentiable at 0 we solve completely the problem determining explicitly the optimal trajectory.

3.1 Lyapunov function

We remark that the Lyapunov technique we use is somewhat non-standard since the energy does not increase at every stage but every N time steps, where N is the least common multiple of all the maturity ages n_i. We could recover a standard Lyapunov function by considering the sum or the maximum over N consecutive periods, however this would make the arguments unnecessarily obscure.
Let us define the function $\Phi : \Delta^0 \to \mathbb{R}$ given by

$$\Phi(\mathbb{X}_0) = G(\mathbb{X}_0) - \sum_{i\in I} \sum_{t=0}^{n_i-2} \tfrac{b^t - b^{n_i-1}}{1-b^{n_i}} U_i(x_t^i) \tag{16}$$

where $G(\mathbb{X}_0)$ is the optimal benefit obtained from state $\mathbb{X}_0$ by using a greedy policy

$$\begin{array}{rcl} G(\mathbb{X}_0) & = & \max \quad \sum_{t=0}^{\infty} b^t \big[\sum_{i=0}^k U_i(x_t^i) - g(c_t) \big] \\ & & \text{s.t. } \boldsymbol{x}^i \in \ell_+^\infty \\ & & \sum_{i=0}^k x_{t+n_i}^i = \sum_{i=0}^k x_t^i \\ & & x_{t+1}^0 = x_t^0 - c_t \end{array}$$

The clue for the subsequent asymptotic analysis is the following property which shows that Φ is a Lyapunov function.

Theorem 13 *Let N be the least common multiple of $\{n_i, i \in I\}$. If $\mathbb{X}_0 \in \Delta^g$ then*

$$\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0) + \sum_{j=0}^{N-1} g(c_j)$$

with strict inequality unless $\mathbb{X}_0 \in \Delta^p$. Hence Φ is a Lyapunov function modulo N.

Proof. To simplify the notation we set $U_t^i = U_i(x_t^i), G_t = G(\mathbb{X}_t)$, and we denote

$$P_t = \sum_{i=0}^k \tfrac{1}{1-b^{n_i}} \sum_{j=t}^{t+n_i-1} b^{j-t} U_j^i.$$

the benefit of a GPC started from state $\mathbb{X}_t$. Since G_t is the optimal benefit we have $G_t \geq P_t$ which can be written as $(1-b^N)G_t \geq (1-b^N)P_t$ and then

$$G_t \geq b^N G_t + \sum_{i=0}^k \tfrac{1-b^N}{1-b^{n_i}} \sum_{j=t}^{t+n_i-1} b^{j-t} U_j^i.$$

Now, Bellman's principle of dynamic programming gives

$$\begin{array}{rcl} G_0 & = & \sum_{j=0}^{t-1} b^j \big[\sum_{i=0}^k U_j^i - g(c_j) \big] + b^t G_t \\ G_t & = & \sum_{j=t}^{N-1} b^{j-t} \big[\sum_{i=0}^k U_j^i - g(c_j) \big] + b^{N-t} G_N \end{array}$$

which plugged into the previous inequality yields

$$\begin{array}{l} b^{N-t} G_N + \sum_{j=t}^{N-1} b^{j-t} \Big[\sum_{i=0}^k U_j^i - g(c_j) \Big] \geq \\ \qquad b^{N-t} G_0 - \sum_{j=0}^{t-1} b^{N-t+j} \Big[\sum_{i=0}^k U_j^i - g(c_j) \Big] + \sum_{i=0}^k \tfrac{1-b^N}{1-b^{n_i}} \sum_{j=t}^{t+n_i-1} b^{j-t} U_j^i. \end{array}$$

Adding up these equations for $t = 0, 1, \ldots, N-1$ we get

$$\begin{array}{l} \tfrac{b(1-b^N)}{1-b} G_N + \displaystyle\sum_{t=0}^{N-1} \sum_{j=t}^{N-1} b^{j-t} \Big[\sum_{i=0}^{k} U_j^i - g(c_j) \Big] \geq \\ \tfrac{b(1-b^N)}{1-b} G_0 - \displaystyle\sum_{t=0}^{N-1} \sum_{j=0}^{t-1} b^{N-t+j} \Big[\sum_{i=0}^{k} U_j^i - g(c_j) \Big] + \sum_{t=0}^{N-1} \sum_{i=0}^{k} \tfrac{1-b^N}{1-b^{n_i}} \sum_{j=t}^{t+n_i-1} b^{j-t} U_j^i \end{array} \tag{17}$$

and using Fubini's rule to exchange all these multiple sums we deduce

$$\tfrac{b(1-b^N)}{1-b} G_N + \sum_{j=0}^{N-1} \tfrac{1-b^{j+1}}{1-b} \left[\sum_{i=0}^{k} U_j^i - g(c_j) \right] \quad \geq \quad \tfrac{b(1-b^N)}{1-b} G_0 - \sum_{j=0}^{N-1} \tfrac{b^{j+1}-b^N}{1-b} \left[\sum_{i=0}^{k} U_j^i - g(c_j) \right]$$

$$+ \sum_{i=0}^{k} \tfrac{1-b^N}{1-b^{n_i}} \left[\sum_{j=0}^{n_i-2} \tfrac{1-b^{j+1}}{1-b} U_j^i + \sum_{j=n_i-1}^{N-1} \tfrac{1-b^{n_i}}{1-b} U_j^i + \sum_{j=N}^{N+n_i-2} \tfrac{b^{j-N+1}-b^{n_i}}{1-b} U_j^i \right]$$

The two sums on the first line may be combined and factored by the term $\frac{1-b^N}{1-b}$ which may then be dropped throughout in order to get

$$b\, G_N \geq b\, G_0 - \sum_{j=0}^{N-1} \sum_{i=0}^{k} U_j^i + \sum_{j=0}^{N-1} g(c_j) + \sum_{i=0}^{k} \left[\sum_{j=0}^{n_i-2} \tfrac{1-b^{j+1}}{1-b^{n_i}} U_j^i + \sum_{j=n_i-1}^{N-1} \tfrac{1-b^{n_i}}{1-b^{n_i}} U_j^i + \sum_{j=N}^{N+n_i-2} \tfrac{b^{j-N+1}-b^{n_i}}{1-b^{n_i}} U_j^i \right]$$

We may now change the order of summation of the first sum and cancel out the terms to deduce

$$b\, G_N \geq b\, G_0 + \sum_{j=0}^{N-1} g(c_j) + \sum_{i=0}^{k} \left[\sum_{j=0}^{n_i-2} \tfrac{b^{n_i}-b^{j+1}}{1-b^{n_i}} U_j^i + \sum_{j=N}^{N+n_i-2} \tfrac{b^{j-N+1}-b^{n_i}}{1-b^{n_i}} U_j^i \right]$$

and dividing by b, rearranging terms and taking into consideration that $n_0 = 1$ we have

$$\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0) + \textstyle\sum_{j=0}^{N-1} g(c_j). \tag{18}$$

When $\mathbb{X}_0 \notin \Delta^p$ we have $G_0 > P_0$ and therefore the inequality (17) as well as its consequence (18) become strict. ■

Corollary 14 *If* ${\gamma_-}^2 + {\gamma_+}^2 > 0$ *then along any optimal trajectory* $\lim_{t\to\infty} c_t = 0$ *and* $\lim_{t\to\infty} x_t^0$ *exists.*

Proof. First observe that Φ is bounded from above by some constant M. Then, proceeding by induction in (18) we obtain

$$M \geq \Phi(\mathbb{X}_{kN}) \geq \Phi(\mathbb{X}_0) + \sum_{j=0}^{kN-1} g(c_j)$$

from which it follows that

$$M \geq \sum_{j=0}^{\infty} g(c_j) = \gamma_- \sum_{c_j<0} c_j + \gamma_+ \sum_{c_j>0} c_j.$$

Let us suppose that $\gamma_+ > 0$. This readily implies that $\frac{M}{\gamma_+} \geq \sum_{c_j>0} c_j \geq 0$.

Using that $x_t^0 = x_0^0 + \sum_{j=0}^{t-1} c_j \in [0, S]$ for all t we can easily deduce that $0 \leq x_0^0 + \sum_{c_j<0} c_j + \sum_{c_j>0} c_j \leq S$ and then there is some constant M' such that $M' \leq \sum_{c_j<0} c_j \leq 0$.

As the two sums are bounded it follows that $\sum_{j=0}^{\infty} |c_j| < \infty$ and therefore $c_j \to 0$. Moreover $\{x_t^0\}$ is a Cauchy sequence since $|x_t^0 - x_{t+k}^0| \leq \sum_{j=t}^{t+k-1} |c_j|$, so the unused area x_t^0 has a limit.

If $\gamma_+ = 0$ and $\gamma_- > 0$ the proof follows the same line. ■

3.2 Asymptotic convergence for greedy trajectories

Since Δ^g is forward invariant under an optimal greedy strategy in the sense that $\mathbb{X}_0 \in \Delta^g$ implies $\mathbb{X}_t \in \Delta^g$ for all $t \geq 0$, it follows that $\Phi(\mathbb{X}_{t+N}) \geq \Phi(\mathbb{X}_t)$ showing that an optimal sequence of states $\mathbb{X}_t$ becomes an "N-step" uphill strategy for the Lyapunov function Φ as soon as it enters the set Δ^g. This property is used to prove that we have convergence to Δ^p. The proof is valid only when the optimal trajectory becomes greedy after finitely many steps. At the end of this subsection, we comment some conditions under which this is assured.

Theorem 15 *Let $\mathbb{X}_0 \in \Delta$ be such that the optimal trajectory satisfies $\mathbb{X}_t \in \Delta^g$ for some $t \in \mathbb{N}$. Then the optimal trajectory converges to a GPC in the sense that*

$$\lim_{t\to\infty} \operatorname{dist}(\mathbb{X}_t, \Delta^p) = 0.$$

We refer the reader to the proof presented in [2, Theorem 4.5]. In the proof the trajectory's uniqueness is key, which forces the restriction of this study into three particular cases: greedy policies and/or strict concave U_0 or linear U_0.

Thanks to Corollary 14 we know that given $\mathbb{X}_0 \in \Delta$ the free land x_t^0 converges along the optimal trajectory. This allows us to state a sharpened version of the theorem above.

Corollary 16 *Let $\mathbb{X}_0 \in \Delta$ be such that the optimal trajectory satisfies $\mathbb{X}_t \in \Delta^g$ for some $t \in \mathbb{N}$. Let $a_\infty = S - x_\infty^0$ where $x_\infty^0 = \lim_{t\to\infty} x_t^0$. Then*

$$\lim_{t\to\infty} \operatorname{dist}\big(\mathbb{X}_t, \Delta^p(a_\infty)\big) = 0.$$

If in addition m.c.d.$\{n_i : i \in I_{a_\infty}^*\} = 1$, then the forest converges to the sustainable state

$$\lim_{t\to\infty} \mathbb{X}_t = \mathbb{X}_{a_\infty}^*. \tag{19}$$

Proof. We treat first the case where ${\gamma_-}^2 + {\gamma_+}^2 > 0$. Let $\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i, \boldsymbol{c}$ be an optimal trajectory for $P(\mathbb{X}_0)$. Given any $\epsilon > 0$ we may find T such that $\sum_{t=T}^{\infty} c_t \leq \frac{\epsilon}{2}$ and then

$$\operatorname{dist}\big(\mathbb{X}_t, \cup_{|a-a_\infty|>\epsilon}\Delta^p(a)\big) > \tfrac{\epsilon}{2} \quad \forall t \geq T.$$

Hence,

$$\operatorname{dist}(\mathbb{X}_t, \Delta^p) \to 0 \iff \operatorname{dist}(\mathbb{X}_t, \cup_{|a-a_\infty|\leq\epsilon}\Delta^p(a)) \to 0$$

and letting $\epsilon \to 0$ we get $\operatorname{dist}(\mathbb{X}_t, \Delta^p(a_\infty)) \to 0$. This result together with Proposition 12 readily imply (19).

If $\gamma_- = \gamma_+ = 0$ then from the definitions of $\underline{a}$ and $\overline{a}$ we have $\underline{a} = \overline{a} = a^\infty$, $\Delta^* = \{\mathbb{X}_{a^\infty}^*\}$ and $\Delta^p = \Delta_{a^\infty}^p$, hence the corollary coincides with the theorem. ■

In fact, a stronger result can be proved when g is differentiable at 0 ($\gamma_- = \gamma_+ = 0$) as it is for example the case when there are no transaction costs ($g \equiv 0$).

Corollary 17 *Let g be differentiable at 0. Let $\mathbb{X}_0 \in \Delta$ be such that the optimal trajectory satisfies $\mathbb{X}_t \in \Delta^g$ for some $t \in \mathbb{N}$. Then there is only one surface, a^*, admitting a sustainable state. If $a^* < S$ then*

$$\lim_{t\to\infty} \mathbb{X}_t = \mathbb{X}_{a^*}^*$$

Proof. It follows easily that there is a unique area $a^* = \underline{a} = \overline{a}$ which gives a sustainable state $\Delta^* = \{\mathbb{X}_{a^*}^*\}$. Moreover, if $a^* < S$ then $\mathbb{X}_t$ converges towards $\mathbb{X}_{a^*}^*$ since $\Delta^p = \{\mathbb{X}_{a^*}^*\}$. The latter follows since conditions

(8)(c) and (d) imply $\sigma_0 U_0'(x^0) = \sigma_i U_i'(x_t^i)$ for all $x_t^i > 0$ when $a^* \in (0, S)$ and we already know that $\Delta^p(0) = \{\mathbb{X}_0^*\}$ [7]. ■

The previous results are valid when the optimal trajectory is greedy or becomes greedy after finitely many steps. In [2], it is proved that this is the case if there is an annual forest species or when $U_i'(S) \geq bU_i'(0)$ for every $i \in I$. These two sufficient conditions are still valid for our model.

3.3 Asymptotic convergence when U_0 is strictly concave

The uniqueness of the optimal trajectory plays a fundamental role in the proof of Theorem 15. If U_0 is strictly concave, then the area balance allows having up to one utility function merely concave non-decreasing without loosing uniqueness of the harvests. We will use this observation to extend the result from greedy to general optimal trajectories.

Theorem 18 *Let $\mathbb{X}_0 \in \Delta$. Let $a_\infty = S - x_\infty^0$ where $x_\infty^0 = \lim_{t\to\infty} x_t^0$. Then*

$$\lim_{t\to\infty} \operatorname{dist}\big(\mathbb{X}_t, \Delta^p(a_\infty)\big) = 0.$$

If in addition m.c.d.$\{n_i : i \in I_{a_\infty}^*\} = 1$, then the forest converges to the sustainable state

$$\lim_{t\to\infty} \mathbb{X}_t = \mathbb{X}_{a_\infty}^*.$$

Proof. Clearly the proof is identical to that of Corollary 16 as long as we can state

$$\lim_{t\to\infty} \operatorname{dist}(\mathbb{X}_t, \Delta^p) = 0. \tag{20}$$

To see (20) we adapt the proof of [2, Theorem 4.7]: every optimal trajectory has an equivalent greedy trajectory in an augmented problem where a dummy variable corresponding to *"bare land"* is added. This new variable must not be confused with the unused land x^0. The bare land is land that belongs to the forest owner but is neither allocated to any forest species nor sold, and hence its benefit function is null. This variable gives an extra degree of freedom allowing us to eliminate every overmature tree after some finite time, so that the augmented equivalent trajectory becomes greedy and we may apply the convergence result for this particular case, from which the conclusion in the general case follows. We obtain as a by product of the proof that the optimal trajectory is *asymptotically greedy*, i.e., $\lim_{t\to\infty} \bar{x}_t^i = 0$. For a detailed proof we refer to [2, Theorem 4.7]. ■

4 Asymptotic convergence when U_0 is linear

In this section we turn to the situation where the price of the land is constant $\rho(a) \equiv \bar{\rho}$. In this case we have a constant sale price $q = \bar{\rho} + \gamma_-$ and a constant buy price $p = \bar{\rho} + \gamma_+$ so that

$$W(a, c) = \begin{cases} -pc & c \geq 0 \\ -qc & c < 0 \end{cases}$$

We also make the assumption that a_t is always strictly less than S, as well as $\overline{a}$ which implies $r_{\overline{a}} = q$. Observe also that $p \geq r_{\underline{a}}$ with equality if $\underline{a} > 0$. We prove that there is no trading of land after time stage $2\max_i n_i$ and state the problem as a simpler fixed land problem for $t \geq 2\max_i n_i$.

[7] An alternative way to reach this conclusion is to observe that $P(\mathbb{X}_0)$ can be seen as an instance of the problem studied in [2], by interpreting the unused land as another species with a differentiable benefit function. Under this interpretation the condition $a^* < S$ means that species 0 is present at the sustainable state and, since its maturity age is equal to one, the results in [2] imply the convergence to the sustainable state as well as the greediness of the optimal trajectory for all $t \geq 2N$.

Remark 2 *If $c_t = 0$ after finitely many steps, then Theorem 18 holds and there is convergence to Δ^p. Notice that we recover the fixed surface problem studied in [2].*

We begin with some technical results needed in the proof of our claim.

Lemma 19 *For all $i \in I$ we have*

(a) if $c_T > 0$ and $x^i_{T+n_i} > 0$ then $\bar{x}^i_{T+n_i+1} = 0$,

(b) if $c_T < 0$ and $T \geq n_i$ then $\bar{x}^i_{T+n_i} = 0$.

Proof. The proof is somewhat involved but the idea is simple: if there is a fraction of land occupied by trees which are not really needed, $\bar{x}^i_{T+n_i+1} > 0$ or $\bar{x}^i_{T+n_i} > 0$, we sell it at some previous stage and buy it back just in time to keep harvests unchanged, and then we compare the optimal benefit with the one obtained with the new trajectory to obtain a contradiction.
(a) Assume by contradiction that $\bar{x}^i_{T+n_i+1} > 0$. Then for $\epsilon > 0$ small enough the following alternative trajectory is feasible

$$\begin{array}{rclrclrcl} \tilde{c}_T &=& c_T - \epsilon & \tilde{x}^i_{T+n_i} &=& x^i_{T+n_i} - \epsilon & \tilde{\bar{x}}^i_{T+n_i+1} &=& \bar{x}^i_{T+n_i+1} - \epsilon \\ \tilde{c}_{T+1} &=& c_{T+1} + \epsilon & \tilde{x}^i_{T+n_i+1} &=& x^i_{T+n_i+1} + \epsilon & && \end{array}$$

Depending on the sign of c_{T+1} the difference between the benefit $\tilde{V}$ of the alternative trajectory and the optimal benefit V is

$$\begin{array}{l} c_{T+1} \geq 0 \quad \Rightarrow \quad \tilde{V} - V = b^T \epsilon p - b^{T+1} \epsilon p = b^T (1-b) \epsilon p > 0 \\ c_{T+1} < 0 \quad \Rightarrow \quad \tilde{V} - V = b^T \epsilon p - b^{T+1} \epsilon q = b^T \epsilon (p - bq) > 0 \end{array}$$

thus, in both cases $\tilde{V} > V$ which contradicts optimality.
(b) Assume that $\bar{x}^i_{T+n_i} > 0$. First note that in each interval of length n_i such as $p+1, \ldots, p+n_i$ there is at least one $\bar{x}^i_t = 0$. Indeed, if this was not the case then at time p we could harvest a small additional area $\epsilon > 0$ and resow it immediately as species i, modifying the trajectory as $\bar{x}^i_t - \epsilon$ for $t = p+1, \ldots, p+n_i$ and $x^i_{p+n_i} + \epsilon$, after which we rejoin the original optimal strategy. This modified trajectory would increase the benefit by an amount $b^p[U_i(u^i_p + \epsilon) - U_i(u^i_p)] > 0$ contradicting optimality. This immediately implies that there is $x^i_{T+l} > 0$ with $l \in [j, n_i - 1]$. Now, let l be the largest index in $[0, n_i - 1]$ such that $x^i_{T+l} > 0$ and consider

$$\begin{array}{rclrclrcl} \tilde{c}_{T-n_i+l} &=& c_{T-n_i+l} - \epsilon & \tilde{x}^i_{T+l} &=& x^i_{T+l} - \epsilon & \tilde{\bar{x}}^i_{T+j} &=& \bar{x}^i_{T+j} - \epsilon, \ j \in [l+1, n_i] \\ \tilde{c}_T &=& c_T + \epsilon & \tilde{x}^i_{T+n_i} &=& x^i_{T+n_i} + \epsilon & && \end{array}$$

Depending on the sign of c_{T-n_i+l} the difference between the benefit $\tilde{V}$ of the alternative trajectory and the optimal value V may be

$$\begin{array}{l} c_{T-n_i+l} > 0 \quad \Rightarrow \quad \tilde{V} - V = b^{T-n_i+l} \epsilon p - b^T \epsilon q = b^{T-n_i+l}(p - b^{n_i - l} q)\epsilon > 0 \\ c_{T-n_i+l} \leq 0 \quad \Rightarrow \quad \tilde{V} - V = b^{T-n_i+l} \epsilon q - b^T \epsilon q = b^{T-n_i+l}(1 - b^{n_i - l})\epsilon q > 0. \end{array}$$

In both cases $\tilde{V} > V$ which contradicts optimality. ∎

Proposition 20 *For all $i \in I$ we have*

(a) if $c_T > 0$ then $x^{*i}_{\underline{a}} \geq x^i_{T+n_i}$,

(b) if $c_T < 0$ and $T \geq n_i$ then $x^{*i}_{\bar{a}} \leq x^i_{T+n_i}$.

Proof. (a) The statement is trivial if $x^i_{T+n_i}=0$. If $x^i_{T+n_i}>0$ we consider an alternative trajectory that postpones n_i periods the purchase of an ϵ of land

$$\begin{array}{rclcrcl}\tilde{c}_T &=& c_T-\epsilon & \quad & \tilde{x}^i_{T+n_i} &=& x^i_{T+n_i}-\epsilon \\ \tilde{c}_{T+n_i} &=& c_{T+n_i}+\epsilon & & \tilde{u}^i_{T+n_i} &=& u^i_{T+n_i}-\epsilon\end{array}$$

From Lemma 19 (a) we know that $u^i_{T+n_i}=\bar{x}^i_{T+n_i}+x^i_{T+n_i}\geq x^i_{T+n_i}$, hence it is enough to take $0<\epsilon<x^i_{T+n_i}$ to assure feasibility of the proposed trajectory and we get

$$\begin{array}{lll}\text{if } c_{T+n_i}\geq 0 & \Rightarrow & 0\geq \frac{\tilde{V}-V}{b^T}=\epsilon p(1-b^{n_i})+b^{n_i}\left[U_i(u^i_{T+n_i}-\epsilon)-U_i(u^i_{T+n_i})\right] \\ \text{if } c_{T+n_i}< 0 & \Rightarrow & 0\geq \frac{\tilde{V}-V}{b^T}=\epsilon p+b^{n_i}\left[-\epsilon q+U_i(u^i_{T+n_i}-\epsilon)-U_i(u^i_{T+n_i})\right] \\ & & \qquad\qquad > \epsilon p(1-b^{n_i})+b^{n_i}\left[U_i(u^i_{T+n_i}-\epsilon)-U_i(u^i_{T+n_i})\right]\end{array}$$

Dividing by $\epsilon>0$ and letting $\epsilon\to 0$ we get $\sigma_i U_i'(u^i_{T+n_i})\geq p\geq r_{\underline{a}}\geq \sigma_i U_i'(x^{*i}_{\underline{a}})$ which leads to $x^{*i}_{\underline{a}}\geq u^i_{T+n_i}$ and the proof follows because $u^i_{T+n_i}\geq x^i_{T+n_i}$.

(b) Here, the statement is trivial for $i\notin I^*_{\overline{a}}$. We consider an alternative trajectory that postpones the selling of the land. The proof is almost identical to that of (a), taking

$$\begin{array}{rclcrcl}\tilde{c}_T &=& c_T+\epsilon & \quad & \tilde{x}^i_{T+n_i} &=& x^i_{T+n_i}+\epsilon \\ \tilde{c}_{T+n_i} &=& c_{T+n_i}-\epsilon & & \tilde{u}^i_{T+n_i} &=& u^i_{T+n_i}+\epsilon\end{array}$$

We conclude easily that $\sigma_i U_i'(u^i_{T+n_i})\leq q=r_{\overline{a}}=\sigma_i U_i'(x^{*i}_{\overline{a}})$ for all $i\in I^*_{\overline{a}}$ that shows $x^{*i}_{\overline{a}}\leq u^i_{T+n_i}$ and the proposition follows from Lemma 19 (b). ■

Proposition 21 *For all $i\in I$ we have*

(a) *if $c_T>0$ and $T\geq n_i$ then $u^i_T\geq x^{*i}_{\underline{a}}$,*

(b) *if $c_T<0$ and $T\geq 2n_i$ then $u^i_T\leq x^{*i}_{\overline{a}}$.*

Proof. (a) The proposition is evident when $\underline{a}=0$. If $\underline{a}>0$, consider an alternative trajectory that brings forward the buying of an ϵ of land

$$\begin{array}{rclcrcl}\tilde{c}_{T-n_i} &=& c_{T-n_i}+\epsilon & \quad & \tilde{x}^i_T &=& x^i_T+\epsilon \\ \tilde{c}_T &=& c_T-\epsilon & & \tilde{u}^i_T &=& u^i_T+\epsilon\end{array} \tag{21}$$

and the benefit difference fulfills

$$0\geq \frac{\tilde{V}-V}{b^T}\geq -b^{-n_i}\epsilon p+U_i(u^i_T+\epsilon)-U_i(u^i_T)+\epsilon p$$

Dividing by $\epsilon>0$ and letting $\epsilon\to 0$ we get $p\geq \sigma_i U_i'(u^i_T)$ which implies $x^{*i}_{\underline{a}}\leq u^i_T$ since $p=r_{\underline{a}}$.

(b) The statement is trivial if $u^i_T=0$. If $u^i_T>0$ and $T\geq 2n_i$, there must be $l\in[1,n_i]$ such that $x^i_{T-n_i+l}>0$ as follows analogously to the proof of Lemma 19 (b). Let l be the greatest index in $[1,n_i]$ such that $x^i_{T-n_i+l}>0$ and take

$$\begin{array}{rclcrcl}\tilde{c}_{T-2n_i+l} &=& c_{T-2n_i+l}-\epsilon, & \quad & \tilde{x}^i_{T-n_i+l} &=& x^i_{T-n_i+l}-\epsilon \\ \tilde{c}_T &=& c_T+\epsilon, & & \tilde{u}^i_T &=& u^i_T-\epsilon \\ \tilde{\bar{x}}^i_{T-n_i+j} &=& \bar{x}^i_{T-n_i+j}-\epsilon,\ j\in[l,n_i-1] & & & & \end{array} \tag{22}$$

and again the proof follows the same lines of (a). ■

Combining all these results we conclude easily that after $2\max_i n_i$ periods there is no buying nor selling of land. That is to say, along an optimal trajectory all the transactions of land should occur during an initial period of length at most $2\max_i n_i$.

Theorem 22 *Along any optimal trajectory, $c_t = 0$ for all $t \geq 2\max_i n_i$.*

Proof. Suppose first that there is $T \geq \max_i n_i$ such that $c_T > 0$. By Proposition (21)(a) we know that $u^i_T \geq x^{*i}_{\underline{a}}$ for all i. The balance of area at stage T tells us

$$\textstyle\sum_i x^i_{T+n_i} = \sum_i u^i_T + c_T > \sum_i x^{*i}_{\underline{a}}$$

while Proposition (20)(a) gives $x^i_{T+n_i} \leq x^{*i}_{\underline{a}}$ for all i which gives a contradiction.

A similar procedure tells us that having $c_T < 0$ with $T \geq 2\max n_i$ is also absurd. ■

4.1 Linear land price and differentiable g

If in particular we have a constant function $\rho(a) = \bar{\rho}$ and $\gamma_- = \gamma_+ = 0$, ($p = q = \bar{\rho}$) we can prove that convergence to the sustainable state occurs in finite time and therefore the problem becomes finite dimensional and the optimal trajectory can be trivially found.

Proposition 23 *For each optimal trajectory we have $x^i_t = u^i_t = x^{*i}_{a^*}$ for all $t \geq 3\max_i n_i$* [8].

Proof. We start by characterizing the optimal harvest: we claim that $u^i_t = x^{*i}_{a^*}$ for $t \geq 2n_i$. To prove this claim, it suffices to repeat the proof of Proposition 21. Notice that since $p = q$ no conditions are required on c_T. Using the trajectory (21) we get

$$0 \geq \tfrac{\tilde{V}-V}{b^T} = (1 - b^{-n_i})\epsilon\bar{\rho} + U_i(u^i_T + \epsilon) - U_i(u^i_T) \Rightarrow \bar{\rho} \geq \sigma_i U'_i(u^i_T) \iff x^{*i}_{a^*} \leq u^i_T$$

while the trajectory (22) gives us

$$0 \geq \tfrac{\tilde{V}-V}{b^T} = (1 - b^{-2n_i+l})\epsilon\bar{\rho} + U_i(u^i_T - \epsilon) - U_i(u^i_T) \Rightarrow \bar{\rho} \leq \sigma_i U'_i(u^i_T) \iff x^{*i}_{a^*} \geq u^i_T$$

thus we conclude $u^i_t = x^{*i}_{a^*}$ for all $t \geq 2n_i$.

From now on we take $t \geq 2\max_i n_i$. Adding up (2) for $i \in I$ we get

$$\textstyle\sum_{i\in I} \bar{x}^i_{t+n_i+1} = \sum_{i\in I} \bar{x}^i_{t+n_i} + \sum_{i\in I} x^i_{t+n_i} - \sum_{i\in I} u^i_{t+n_i}$$

so that using the area constraint (3) and remembering that $c_t = 0$ we can deduce

$$\textstyle\sum_{i\in I} \bar{x}^i_{t+n_i+1} = \sum_{i\in I} \bar{x}^i_{t+n_i} + \sum_{i\in I} u^i_t - \sum_{i\in I} u^i_{t+n_i}$$

and since $u^i_t = x^{*i}_{a^*} = u^i_{t+n_i}$ we get $\sum_{i\in I} \bar{x}^i_{t+n_i+1} = \sum_{i\in I} \bar{x}^i_{t+n_i}$. However, we know from Theorem 18 that $\bar{x}^i_t \to 0$, so the only possibility is that $\sum_{i\in I} \bar{x}^i_{t+n_i} = 0$ for all $t \geq 2\max_i n_i$, which then gives $\bar{x}^i_t = 0$ for $t \geq 3\max_i n_i$. Finally, this implies $x^i_t = u^i_t = x^{*i}_{a^*}$ for $t \geq 3\max_i n_i$ as was to be proved. ■

5 Land conversion costs

The results presented up to now can be applied to a broader class of problems, like those of the forests where land conversion is costly. In our case, this cost is $g(x^0_t - x^0_{t+1})$ when we buy or sell land, that is to say when we change the use of the land from unused to a forestry use or viceversa. In this section we aim to extend these results to the case where the land conversion from any species to another could be costly. This problem was studied by Salo and Tahvonen [16] for the case of a one species forest and an alternative annual use of the land. They showed that when land conversion costs are introduced, new optimal periodic cycles appear.

[8] In Corollary 17 we saw that if g is differentiable at 0 then $a^* = \underline{a} = \overline{a}$.

This suggests that taking them into consideration will make our set Δ^p bigger, and that there may be more sustainable states. We model the conversion costs as a convex function g that depends only on harvests and sows at each stage t, and fulfills the following two properties

$$\begin{array}{rl} i) & g(u_t^1,\ldots,u_t^k,x_{t+n_1}^1,\ldots,x_{t+n_k}^k) \geq 0 \\ ii) & g(u_t^1,\ldots,u_t^k,u_t^1,\ldots,u_t^k) = 0\,, \end{array} \tag{23}$$

i.e., the conversion costs is always non-negative and it is null when there is no land conversion. The problem is now

$$P_g(\mathbb{X}_0)\;\begin{cases} \text{maximize} & \sum_{t=0}^{\infty} b^t\big[\sum_{i\in I} U_i(u_t^i) - g\big((u_t^i)_{i\in I},(x_{t+n_i}^i)_{i\in I}\big)\big] \\ \text{subject to} & (2) \text{ and } (3) \\ & \boldsymbol{x}^i,\bar{\boldsymbol{x}}^i,\boldsymbol{u}^i \in \ell_+^\infty \text{ with } \mathbb{X}_0 \text{ given.} \end{cases}$$

The characterization of Δ^*, Δ^p and Δ^g is more involved than before, depending strongly on the particular form of the function g. However, it is worth mentioning that under conditions (23) the function Φ defined by (16) where $G(\mathbb{X}_0)$ stands for the optimal greedy value of problem $P_g(\mathbb{X}_0)$ remains a Lyapunov function.

Theorem 24 *Let N be the least common multiple of $\{n_i, i\in I\}$. If $\mathbb{X}_0 \in \Delta^g$ then*

$$\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0) + \textstyle\sum_{j=0}^{N-1} g(u_j^1,\ldots,u_j^k,x_{j+n_1}^1,\ldots,x_{j+n_k}^k)$$

with strict inequality unless $\mathbb{X}_0 \in \Delta^p$, so that Φ is a Lyapunov function modulo N.

Proof. We can repeat the steps of the proof of Theorem (16), where the Bellman's principle of dynamic programming is now stated as

$$\begin{array}{rcl} G_0 &=& \sum_{j=0}^{t-1} b^j\big[\sum_{i=0}^{k} U_j^i - g(u_j^1,\ldots,u_j^k,x_{j+n_1}^1,\ldots,x_{j+n_k}^k)\big] + b^t G_t \\ G_t &=& \sum_{j=t}^{N-1} b^{j-t}\big[\sum_{i=0}^{k} U_j^i - g(u_j^1,\ldots,u_j^k,x_{j+n_1}^1,\ldots,x_{j+n_k}^k)\big] + b^{N-t} G_N \end{array}$$

We continue in the same manner and instead of (18) we retrieve

$$\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0) + \textstyle\sum_{j=0}^{N-1} g(u_j^1,\ldots,u_j^k,x_{j+n_1}^1,\ldots,x_{j+n_k}^k)$$

the second term of the rhs is always positive due to $i)$, so cancelling out we conclude as before. ■

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