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About optimal harvesting policies for a multiple species forest without discounting

Adriana Piazza

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Abstract We study the optimal harvesting of a mixed forest composed of multiple species, each one having a different maturity age, where only mature trees can be harvested. We prove the existence of an optimal program and the equivalence of maximal, optimal and minimal value-loss programs. We characterize the unique *golden rule stock* and prove that it is *sustainable*, i.e., it is invariant along the optimal program. Furthermore, we also prove that along *any* good program from any initial condition there is convergence of the forest's state to this sustainable state. Finally, we define a value function in the set of forest states and define a pre-order that provides an alternative way of characterizing the golden rule stock and may potentially have independent interest.

Keywords Existence of optimal programs · Golden-rule stock · Maximal, optimal and minimal value loss programs · Asymptotic stability · Forest management · Environmental economics

JEL Classification C62 · D90 · Q23

1 Introduction

In Samuelson (1976), the author explores whether maximal sustainable harvest yield is obtained if a forest plantation is managed aiming for maximal economic benefit. Samuelson focused the optimal harvesting problem from the point of view of competition and price theory of the firm and, perhaps, the modern interest in the economics

A. Piazza (✉)
Departamento de Matemática, Universidad Técnica Federico Santa María,
Avda. España 1680, Casilla 110-V, Valparaíso, Chile
e-mail: adriana.piazza@usm.cl

of forestry can be dated to that moment. It was necessary to wait until the papers of Mitra and Wan (1985, 1986), to have the problem of the optimal harvesting of a multi-aged single species forest stated in a form fruitful for the application of the general theory of intertemporal allocation. In the undiscounted case, they prove the existence of a maximal program from every initial state following the method developed by Gale (1967), while in the discounted case the existence is a direct consequence of the Weierstrass Theorem. They prove the existence of a state invariant under the optimal policy, the so-called *golden rule stock* or *sustainable state*. They also take particular care in characterizing the asymptotic stability of this stationary state, both in the undiscounted and discounted settings. If the benefit function is linear then the state of the forest describes a periodic trajectory for every initial state and furthermore, the policy function can be explicitly described. The discount factor affects the period of the trajectory. When the benefit function is strictly concave the policy function has not been explicitly found in the general case. Nevertheless, in Mitra and Wan (1986) it is established that when future utilities are not discounted, the state converges asymptotically towards the golden rule stock. The discounted case has proven to be more difficult to solve and its complete characterization remains open.

This last case has been further investigated and several partial results are now known. Among them, we point out the works by Salo and Tahvonen (2002, 2003, 2004) showing that every initial condition close enough to the sustainable state yields a periodic optimal program so that this state is not even a local attractor. The same authors extend the model to include the possibility of allocating some land to an annual alternative use. This new case may be conceived as a particular two-species forest where the second species has a maturity age equal to one. In this setting the authors prove that the sustainable state is a local saddle point, whenever it is optimal to allocate part of the land to the annual use.

The optimal management of a one-species forest with discount is also studied by Rapaport et al. (2003) using a model where only mature trees older than a certain age may be harvested, addressing some of the effects of delay in the management of natural resources. They define a *greedy policy* as one where each tree is harvested as soon as it reaches maturity, showing that every optimal policy becomes greedy and the resulting optimal program becomes periodic after a finite time.

In Cominetti and Piazza (2009), the authors consider a mixed forest composed by several species of different maturity ages, with harvest restricted to mature trees as in the Rapaport et al. model. They prove the existence and uniqueness of a sustainable state and the fact that an optimally managed forest converges towards the set of states for which the corresponding optimal trajectories are periodic. Under a mild additional condition on the maturity ages this implies the convergence of the forest towards the sustainable state.[1] Asymptotic convergence of optimal programs and *turnpike properties*, have been established for wide classes of dynamic optimization problems, most

[1] The model is extended in Piazza (2009) to include a land market, i.e., any fraction of the land can be bought or sold at any time step. The analysis becomes more difficult because the uniqueness of the sustainable state is lost. Nevertheless, the main result is still valid under very general conditions: along any optimal programs, the state converges towards one of the sustainable states or to the set of periodic optimal cycles.

of them issued from the literature on economic growth models. For a survey of the general results available we refer to Arkin and Evstigneev (1979), Dana et al. (2006), Makarov and Rubinov (1977), McKenzie (1986, 2002) as well as Zaslavski (2006). A distinguishing feature in Cominetti and Piazza (2009) is that global convergence is obtained in a discounted utility framework with no restriction on the discount factor, while most convergence or *turnpike* theorems are either local or assume discount factors close to one.

In this article we consider the multiple species model of Cominetti and Piazza (2009) in the undiscounted setting. We find an explicit characterization of the unique *golden rule stock*, defined (as usual) as the solution of the finite dimensional optimization problem $\max \{u(\mathbb{X}, \mathbb{X}) : \text{the transition } \mathbb{X} \to \mathbb{X} \text{ is feasible}\}$, where $u(\mathbb{X}, \mathbb{Y})$ is the benefit obtained today, when the state of the forest today is $\mathbb{X}$ and the state tomorrow is $\mathbb{Y}$. We prove the convergence of *any* good program to the golden rule stock. We use this convergence to prove the existence of an optimal program as well as the equivalence between maximal, optimal and minimal value loss programs. We introduce a *value function*, V, for our model in the undiscounted setting, as it is done first by Brock and Majumdar (1988) and later by Khan and Mitra (2006) and Zaslavski (2007). We endow the set of states with the ad hoc pre-order, $\preceq$, which assures that: $\mathbb{X} \preceq \mathbb{Y} \Rightarrow V(\mathbb{X}) \leq V(\mathbb{Y})$.[2] We show how this pre-order can be used to characterize the golden rule stock[3] and to obtain the value loss function with an alternative argument.

We conclude this introduction by drawing attention to the fact that in a recent paper, (Mitra 2005) has proposed a social welfare criterion, tested it on the Mitra-Wan one-specie forestry model, and found it to be equivalent to the overtaking criterion for this particular model.[4] A next step of the analysis presented here for a multi-species forest would be to see how questions of optimal forest management fare not only in terms of the criterion proposed and studied by Mitra, but also under a larger rubric of "sustainability".

The remainder of the paper is organized as follows. Section 2 introduces the model to be used. In Sect. 3 we prove the existence and uniqueness of a *golden rule stock* and define a *value loss function*. In Sect. 4 we introduce the definition of *good* and *bad* programs and characterize the asymptotic behavior of *good* programs. In the following section we consider the notions of *maximal* and *optimal* programs and prove that in our case they coincide. This allows us to show one of the main results of this work: the existence of an optimal program. This section also includes the existence and uniqueness of a *sustainable state*. Finally, in Sect. 5 we introduce a value function defined on the set of forest states and a pre-order that could be used to derive the results in Sect. 3 and that may have interest on its own.

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2 We observe that, in general, calculating $V(\mathbb{X})$ is not possible. Nevertheless, for some pairs $(\mathbb{X}, \mathbb{Y})$ we can establish whether $V(\mathbb{X}) \leq V(\mathbb{Y})$ without calculating $V(\mathbb{X})$ and $V(\mathbb{Y})$.

3 This alternative definition of the golden rule stock is formally identical to the definition provided in Gale (1967).

4 As is well-known, the problem of providing an axiomatic basis for the overtaking criterion dates to Brock (1970); also see the recent work of Basu and Mitra (2003, 2007). For a general discussion of the interplay between issues of forest management and of intertemporal equity, see Khan (2005) and his references. And for a recent overview statement on the issue of discounting, see Samuelson (2008) and other references in the issue of which it is a part.

2 Model formulation

We consider a discrete time model for the optimal management of a forest of total area 1 occupied by k species $I = \{1, \ldots, k\}$ with maturity ages of $n_1, \ldots, n_k$ years respectively. For each period $t \in \mathbb{N}$ we denote $x_i^j(t) \geq 0$ the area covered by trees of species i that are j years old with $j = 1, \ldots, n_i$ and $\bar{x}_i(t) \geq 0$ the area occupied by over-mature trees (older than n_i). We must decide how much land $u_i(t) \geq 0$ to harvest and how to reallocate this land to new seedlings. Assuming that only mature trees can be harvested we must have $u_i(t) \leq \bar{x}_i(t) + x_i^{n_i}(t)$, and then the area not harvested in that period will comprise the over-mature trees at the next step, namely

$$\bar{x}_i(t+1) \;=\; \bar{x}_i(t) + x_i^{n_i}(t) - u_i(t). \tag{1}$$

The fact that immature trees cannot be harvested is represented by

$$x_i^{j+1}(t) \;=\; x_i^j(t-1) \quad j = 1, \ldots, n_i - 1. \tag{2}$$

The total harvested area $\sum_{i \in I} u_i(t)$ is allocated to new seedlings which is expressed by the equation

$$\sum_{i \in I} x_i^1(t+1) = \sum_{i \in I} u_i(t). \tag{3}$$

A representation of the forest in terms of the age distribution at time t is provided by the *state* $\mathbb{X}(t) = (X_1(t), \ldots, X_k(t))$ where $X_i(t) = \left(x_i^1(t), x_i^2(t), \ldots, x_i^{n_i}(t), \bar{x}_i(t)\right)$ describes the areas occupied in year t by trees of species i with ages $1, 2, \ldots, n_i$ and over n_i. The first and last components of each vector $X_i(t)$ are controlled by the sowing and harvesting policies. Notice that we do not control $\mathbb{X}(0)$ which corresponds to the initial state reflecting the age-class composition of the forest at time $t = 0$.

We denote $\Delta \subset \mathbb{R}_+^N$, $N = \sum_{i \in I}(n_i + 1)$, the set of all initial states $\mathbb{X}$ such that $\sum_{i \in I}[\bar{x}_i + \sum_{j=1}^{n_i} x_i^j] = 1$. Clearly, (1), (2) and (3) imply that $\mathbb{X}(t) \in \Delta$ for all $t \in \mathbb{N}$.

Definition 1 We call $\{\mathbb{X}(t)\}_{t \in \mathbb{N}}$ a *program* if $\mathbb{X}(t) \in \Delta$ and constraints (1) to (3) are satisfied for all t. We call $\{\mathbb{X}(t)\}_{t \in \mathbb{N}}$ a *program from* $\mathbb{X}$ if $\{\mathbb{X}(t)\}_{t \in \mathbb{N}}$ is a program and $\mathbb{X}(0) = \mathbb{X}$.

We will denote a program simply by $\{\mathbb{X}(t)\}$ when no confusion arises. Observe that given $\{\mathbb{X}(t)\}$, Eq. (1) allows us to compute $u_i(t)$ for all $i \in I, t \geq 0$. We point out also that fixing $\mathbb{X}(0)$ and $u_i(t)$ $\forall i, t$ does not determine $\{\mathbb{X}(t)\}$.

Let us define the *transition possibility set* Ω as

$$\Omega = \left\{ (\mathbb{X}, \mathbb{Y}) \in \Delta \times \Delta \Big/ \begin{array}{ll} y_i^{j+1} = x_i^j & \text{for all } j = 1, \ldots, n_i - 1 \\ \bar{x}_i + x_i^{n_i} - \bar{y}_i \geq 0 & \end{array} \right\}$$

We claim that an equivalent definition of a program can be given in terms of Ω:

Remark 1 The sequence $\{\mathbb{X}(t)\}$ is a program iff $(\mathbb{X}(t), \mathbb{X}(t+1)) \in \Omega$ for all $t \geq 0$.

Indeed, it is easy to see that (2) is equivalent to the equation in the definition of Ω. Moreover, re-writting (1) as $u_i(t) = \bar{x}_i(t) + x_i^{n_i}(t) - \bar{x}_i(t+1)$ we see that $u_i(t) \geq 0$ is equivalent to the inequality in the definition. Finally, constraint (3) can be recovered from (1), (2) and the fact that $\mathbb{X}(t)$ and $\mathbb{X}(t+1)$ cover the same area.

To ease the notation we denote $\boldsymbol{u} = (u_1, \ldots, u_k)$. Given $\boldsymbol{u} \in \mathbb{R}_+^k$, let us define as well the set $\Omega_{\boldsymbol{u}} \subseteq \Delta \times \Delta$ as

$$\Omega_{\boldsymbol{u}} = \left\{ (\mathbb{X}, \mathbb{Y}) \in \Omega \Big/ \bar{y}_i = \bar{x}_i + x_i^{n_i} - u_i \right\}$$

i.e., the pair $(\mathbb{X}, \mathbb{Y})$ belongs to $\Omega_{\boldsymbol{u}}$ iff the state $\mathbb{Y}$ can be reached from $\mathbb{X}$ within one time step while harvesting u^i of species i.

Benefit is obtained from the harvests and is represented by the functions U_i, $i \in I$ where $U_i : \mathbb{R}_+ \to \mathbb{R}$ is smooth, increasing and strictly concave for each $i \in I$. Hence, $\sum_{i \in I} U_i(u_i)$ is the benefit obtained today if the forest today is $\mathbb{X}$ and the forest tomorrow is $\mathbb{Y}$ where $(\mathbb{X}, \mathbb{Y}) \in \Omega$ and $u_i = \bar{x}_i + x_i^{n_i} - \bar{y}_i$ for all $i \in I$.

3 A golden-rule stock and the value-loss function

Definition 2 A *golden rule stock* $\hat{\mathbb{X}}$ is a solution to the problem

$$(S) \quad \max_{\mathbb{X}, \boldsymbol{u}} \left\{ \sum_{i \in I} U_i(u_i) \Big/ (\mathbb{X}, \mathbb{X}) \in \Omega_{\boldsymbol{u}} \right\}$$

Remark 2 Observe that condition $(\mathbb{X}, \mathbb{X}) \in \Omega$ implies $x_i^j = x_i$ for all $j = 1, \ldots, n_i$. Moreover, $(\mathbb{X}, \mathbb{X}) \in \Omega$ implies that $x_i = u_i$ for all $i \in I$. These two properties will be useful in the sequel.

Theorem 1 *$\hat{\mathbb{X}}$ is of the form $\hat{X}_i = \hat{u}_i(1, \ldots, 1, 0)$ where $\hat{\boldsymbol{u}}$ is the unique solution to the following strictly concave optimization problem*

$$(S') \quad \max_{\boldsymbol{u} \in \mathbb{R}_+^k} \left\{ \sum_{i \in I} U_i(u_i) \Big/ \sum_{i \in I} n_i u_i \leq S \right\}.$$

Even more, letting $\hat{r}$ be the Lagrange multiplier associated to the area constraint $\sum_{i \in I} n_i u_i \leq S$, the optimal solution is characterized by $U_i'(\hat{u}_i)/n_i \leq \hat{r}$ for all i with $U_i'(\hat{u}_i)/n_i = \hat{r}$ if $\hat{u}_i > 0$.

Proof Let us first observe that both problems (S) and (S') have solutions due to the continuity of the functions U_i and the fact that the corresponding feasible sets are compact. Furthermore, the solution of (S') is unique because the functions U_i are strictly concave.

Thanks to Remark 2 we know that $x_i^j = x_i$ for all $j = 1, \ldots, n_i$ and $u_i = x_i$. It is easy to see that $\bar{x}_i = 0$ because otherwise a state $\tilde{\mathbb{X}}$ with $\tilde{\bar{x}}_i = 0$ and $\tilde{u}_i = \tilde{x}_i^j = x_i + \bar{x}_i/n_i$ would provide a greater value of the objective function of problem (S). Hence, the area constraint can be expressed as $\sum_i n_i u_i = S$ and it can also be relaxed to $\sum_i n_i u_i \leq S$ due to the fact that the functions U_i are increasing. Then finding the solution to (S) can be reduced to solving (S').

Consider the Lagrangian associated to (S'), where we have relaxed the inequality constraint,

$$\mathcal{L} = \sum_{i \in I} U_i(u_i) + r\left(S - \sum_{i \in I} n_i u_i\right). \tag{4}$$

It is easy to see that the Slater regularity condition is satisfied (consider for example the point $u_i = 1/(2kn_i)$) and given the smoothness of the objective function, we can use the Karush-Kuhn-Tucker theorem (see, for example: Borwein and Lewis 2006) to assert that there is a Lagrange multiplier, $\hat{r}$, such that $(\hat{\boldsymbol{u}}, \hat{r})$ is a saddle point of the Lagrangian. This yields

$$\sum_{i \in I} U_i(\hat{u}_i) \geq \sum_{i \in I} U_i(u_i) + \hat{r}(S - \sum_{i \in I} n_i u_i) \tag{5}$$

Furthermore, the pair $(\hat{\boldsymbol{u}}, \hat{r})$ satisfies the corresponding KKT conditions:

$$\frac{\partial \mathcal{L}}{\partial u_i} = U_i'(u_i) - r n_i \geq 0 \tag{6}$$

$$u_i \left(\frac{\partial \mathcal{L}}{\partial u_i}\right) = u_i(U_i'(u_i) - r n_i) = 0. \tag{7}$$

that readily imply $U_i'(\hat{u}_i)/n_i \leq \hat{r}$ for all i and $U_i'(\hat{u}_i)/n_i = \hat{r}$ if $\hat{u}_i > 0$. □

For the rest of this paper we assume without loss of generality that the species are ordered in such a way that $U_1'(0)/n_1 \geq U_2'(0)/n_2 \geq \cdots \geq U_k'(0)/n_k$.

The ordering of the species and the strict concavity of U_i then imply that whenever $\hat{x}_i > 0$ we must also have $\hat{x}_j > 0$ for all $j < i$. This allows us to write the equality $r = U_1'(\hat{u}_1)/n_1$ and also leads to a constructive method for finding the optimal solution of (S') in which the values of $\hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots$ are increased sequentially:

– Increase $\hat{x}_1$ as much as possible until $U_1'(\hat{x}_1)/n_1$ decreases to the value $U_2'(0)/n_2$
– Continue increasing $\hat{x}_1$ and $\hat{x}_2$ simultaneously preserving the equality $U_1'(\hat{x}_1)/n_1 = U_2'(\hat{x}_2)/n_2$ until this common value decreases to the level $U_3'(0)/n_3$.
– Continue this procedure with $\hat{x}_3, \hat{x}_4, \ldots$ stopping as soon as $\sum_i n_i \hat{x}_i = S$.

The point found by this procedure satisfies the optimality conditions for (S') and it is therefore the unique optimal solution.

Definition 3 Let the *value loss function* $\delta : \Omega \to \mathbb{R}$ be defined by

$$\delta(\mathbb{X}, \mathbb{Y}) = \sum_{i \in I} \left[U_i(\hat{x}_i) - U(u_i) + r n_i u_i - r \sum_{j=1}^{n_i} x_i^j \right]$$

where $u_i = \bar{x}_i + x_i^{n_i} - \bar{y}_i$.

We claim that $\delta(\cdot, \cdot) \geq 0$. Indeed, by (5) we have

$$\begin{aligned} \delta(\mathbb{X}, \mathbb{Y}) &= \sum_{i \in I} \left[U_i(\hat{u}_i) - U_i(u_i) + r n_i u_i - r \sum_{j=1}^{n_i} x_i^j \right] \\ &\geq \sum_{i \in I} r \bar{x}_i \geq 0 \qquad \forall (\mathbb{X}, \mathbb{Y}) \in \Omega \end{aligned} \tag{8}$$

4 Good and bad programs

We present the notion of *good* programs originally due to Gale (1967) adapted to our framework and notation. Let $J_T(\{\mathbb{X}(t)\}_{t=0}^T) = \sum_{t=0}^{T-1} \sum_{i \in I} U_i\,(u_i(t)) - U_i(\hat{u}_i)$.

Definition 4 A program $\{\mathbb{X}(t)\}$ is called *good* if there exists $M \in \mathbb{R}$ such that for all $T \geq 0$, $J_T(\{\mathbb{X}(t)\}_{t=0}^T) \geq M$. A program is *bad* if $\lim_{T \to \infty} J_T(\{\mathbb{X}(t)\}_{t=0}^T) = -\infty$.

Definition 5 Given the program $\{\mathbb{X}(t)\}$, we define the *utility of a program* as $J(\{\mathbb{X}(t)\}_{t \in \mathbb{N}}) = \lim_{T \to \infty} J_T(\{\mathbb{X}(t)\}_{t=0}^T)$ whenever the limit exists.

In the sequel, and when no confusion can arise, we will write simply δ_t and J_T and J instead of $\delta\,(\mathbb{X}(t), \mathbb{X}(t+1))$ and $J_T(\{\mathbb{X}(t)\}_{t=0}^T)$ and $J(\{\mathbb{X}(t)\}_{t \in \mathbb{N}})$.

In this section we prove the classical equivalence between the fact that $\{\mathbb{X}(t)\}$ is a good program and the convergence of the corresponding sum of value losses, $\sum_t \delta_t$, that in particular implies the partition of programs into good and bad programs. We also prove that J_T is always bounded above and that it converges when the program is good and that $\lim_{T \to \infty} J_T = -\infty$ otherwise.

Proposition 1 *For any program* $\{\mathbb{X}(t)\}_{t \in \mathbb{N}}$, *we have*

$$J_T = -\sum_{t=0}^{T-1} \delta_t + r \sum_{i \in I} \left\{ n_i [\bar{x}_i(0) - \bar{x}_i(T)] + \sum_{j=2}^{n_i} (j-1)[x_i^j(0) - x_i^j(T)] \right\} \tag{9}$$

There exists $M < \infty$ *such that* $J_T \leq M$ *for all* T. *Furthermore, we have the following equivalence*

i. $\{\mathbb{X}(t)\}$ *is good iff* $\sum_{t=0}^{\infty} \delta_t < \infty$,
ii. $\{\mathbb{X}(t)\}$ *is bad iff* $\sum_{t=0}^{\infty} \delta_t = \infty$.

Proof Using the definition of δ_t we obtain

$$
\begin{aligned}
J_T &= \sum_{t=0}^{T-1}\left\{-\delta_t + r\sum_{i\in I}[n_i u_i(t) - \sum_{j=1}^{n_i} x_i^j(t)]\right\}\\
&= -\sum_{t=0}^{T-1}\delta_t + r\sum_{i\in I}\sum_{t=0}^{T-1}[n_i(\bar{x}_i(t) + x_i^{n_i}(t) - \bar{x}_i(t+1)) - \sum_{j=1}^{n_i} x_i^j(t)]\\
&= -\sum_{t=0}^{T-1}\delta_t + r\sum_{i\in I}\left\{n_i(\bar{x}_i(0)-\bar{x}_i(T)) + \underbrace{\sum_{t=0}^{T-1}[n_i x_i^{n_i}(t) - \sum_{j=1}^{n_i} x_i^j(t)]}_{(A)}\right\}
\end{aligned}
$$

Using (2) the term (A) can be expressed as:

$$
(A) = \sum_{t=0}^{T-1}[n_i x_i^{n_i}(t) - \sum_{j=1}^{n_i} x_i^{n_i}(t+n_i-j)] = \sum_{t=0}^{T-1}[n_i x_i^{n_i}(t) - \sum_{j=t}^{t+n_i-1} x_i^{n_i}(j)],
$$

and then we use Fubini's rule to exchange the order of the multiple sum

$$
\begin{aligned}
(A) &= \sum_{t=0}^{T-1} n_i x_i^{n_i}(t) - \left[\sum_{j=0}^{n_i-2}(j+1)x_i^{n_i}(j) + n_i\sum_{j=n_i-1}^{T-1} x_i^{n_i}(j)\right.\\
&\qquad\left. + \sum_{j=T}^{T+n_i-2}(T-j+n_i-1)x_i^{n_i}(j)\right]\\
&= \sum_{t=0}^{n_i-2}(n_i-t-1)x_i^{n_i}(t) - \sum_{t=T}^{T+n_i-2}(T-t+n_i-1)x_i^{n_i}(t)\\
&= \sum_{j=2}^{n_i}(j-1)x_i^j(0) - \sum_{j=2}^{n_i}(j-1)x_i^j(T) \qquad \text{(using (2))}
\end{aligned}
$$

Substituting this alternative expression of (A) in the previous equation for J_T we deduce (9). Since Δ is compact, there is M such that

$$
-M < r\sum_{i\in I}\left[n_i(\bar{x}_i(0)-\bar{x}_i(T)) + \sum_{j=2}^{n_i}(j-1)(x_i^j(0)-x_i^j(T))\right] < M
$$

for all $\mathbb{X}(0)$ and $\mathbb{X}(T)$ in Δ. Hence we get

$$-\sum_t \delta_t - M < J_T < -\sum_t \delta_t + M \qquad \forall T \tag{10}$$

We know that $\delta_t \geq 0$, thus $J_T \leq M$ for all T.

If $\sum_t \delta_t = \infty$, it is evident that $\lim_T J_T = -\infty$. On the other hand, if $\sum_t \delta_t \in \mathbb{R}$, then $J_T > M' = -\sum_t \delta_t - M$. The equivalence follows from the fact that the sum of positive terms, $\sum_t \delta_t$, can only converge to a real number or diverge to $+\infty$. □

Corollary 1 *Along any program such that* $\mathbb{X}(0) = \mathbb{X}(T)$ *we have* $J_T \leq 0$.

Given any $\mathbb{X}(0) \in \Delta$, there is always at least one good program from it. To see this, it suffices to observe that $\delta(\hat{\mathbb{X}}, \hat{\mathbb{X}}) = 0$ and that from any $\mathbb{X}_0$ there are programs that reach $\hat{\mathbb{X}}$ in finite time and remain there afterwards.[5] We will see immediately that along *any* good program there is asymptotic convergence of the state to $\hat{\mathbb{X}}$.[6]

Theorem 2 *Along any good program we have* $\lim_{t\to\infty} \mathbb{X}(t) = \hat{\mathbb{X}}$.

Proof Let us first prove that $\lim_t \bar{x}_i(t) = 0$ and $\lim_t u_i(t) = \hat{u}_i$.

Since $\sum_{t=0}^{\infty} \delta_t < \infty$, we know that $\lim_{t\to\infty} \delta_t = 0$. Let $\mathbb{X}$ be any accumulation point of $\{\mathbb{X}(t)\}$, and let $\mathbb{Y}$ be such that $(\mathbb{X}, \mathbb{Y})$ is an accumulation point of $(\mathbb{X}(t), \mathbb{X}(t+1))$ (we know that such a $\mathbb{Y}$ exists due to the compacity of Ω). The function $\delta(\cdot, \cdot)$ is continuous, hence $\delta(\mathbb{X}, \mathbb{Y}) = 0$.

From (8), we easily see that $\bar{x}_i = 0$ for all i.

Given that the benefit functions are strictly concave and differentiable, we know that

$$\sum_{i\in I} U_i(\hat{u}_i) - U_i(u_i) + U_i'(\hat{u}_i)(u_i - \hat{u}_i) \geq 0 \tag{11}$$

with equality iff $u_i = \hat{u}_i$ for all $i \in I$. Subtracting $\delta(\mathbb{X}, \mathbb{Y}) = 0$ from this expression we get

$$\begin{aligned} 0 &\leq \sum_{i\in I} \left[U_i'(\hat{u}_i)(u_i - \hat{u}_i) - r n_i u_i + r \sum_j x_i^j \right] \\ &= \sum_{i\in I} \left[(U_i'(\hat{u}_i) - r n_i) u_i - U_i'(\hat{u}_i)\hat{u}_i \right] + r S \qquad (\text{using } \bar{x}_i = 0) \end{aligned}$$

[5] As an example of one of these programs consider the following: let $T = \max\{n_i\}$ and set $\boldsymbol{u}(t) = 0,\ t = 0, \ldots, T-1$. By year T, all the surface is occupied by mature and over-mature trees, and we can harvest $\boldsymbol{u}(t)$ such that $\sum u_i(t) = \sum \hat{u}_i$ and sow exactly $x_i^1(t+1) = \hat{u}_i$ for all $t \geq T$. The state $\hat{\mathbb{X}}$ is reached by $t = 2T$ and $\mathbb{X}(t) = \hat{\mathbb{X}}$ afterwards.

[6] In the literature, this property is sometimes referred as *asymptotic turnpike*, (see McKenzie 1986; Zaslavski 2006).

$$= \sum_{i\in I}(U_i'(\hat{u}_i) - rn_i)u_i - \sum_{i\in I}(U_i'(\hat{u}_i) - rn_i)\hat{u}_i$$

$$= \sum_{i\in I}(U_i'(\hat{u}_i) - rn_i)u_i \qquad \text{(using(7))}$$

$$\leq 0 \qquad \text{(using(6))}.$$

Hence, we have $\sum_{i\in I}(U_i'(\hat{u}_i) - rn_i)u_i = 0$, which implies that (11) holds with equality and consequently $u_i = \hat{u}_i, \forall i$.

In a second place, by considering (1) and letting $t \to \infty$ we get

$$\lim_t x_i^{n_i}(t) = \lim_t \; [\bar{x}_i(t+1) + u_i(t) - \bar{x}_i(t)] = \hat{u}_i$$

and using (2) to we get $x_i^j(t) \to \hat{u}_i$ for all $j = 1, \ldots, n_i$ and of course $\lim_{t\to\infty} \mathbb{X}(t) = \hat{\mathbb{X}}$. □

5 Existence of optimal programs

Following McKenzie (2002, p. 256) we adopt the following definitions,

Definition 6 A program $\mathbb{X}^*(t)$ from $\mathbb{X}_0$ is *optimal* if given any program from $\mathbb{X}_0$ we have

$$\lim_{T\to\infty}\sup \sum_{t=0}^{T}\sum_{i\in I} U_i(u_i(t)) - U(u_i^*(t)) \leq 0$$

A program $\mathbb{X}^*(t)$ from $\mathbb{X}_0$ is *maximal* if for any program from $\mathbb{X}_0$ we have

$$\lim_{T\to\infty}\inf \sum_{t=0}^{T}\sum_{i\in I} U_i(u_i(t)) - U(u_i^*(t)) \leq 0$$

In this section we will prove the existence of at least one optimal program from every initial state. We start by presenting a proof of the existence of a minimal value loss program from any initial state and we then proceed to prove the equivalence between optimal, maximal and minimal value loss programs. These two results together, directly imply the existence of a maximal and optimal program. Finally, we prove that the golden rule stock is stationary under the optimal policy, i.e., it is the *sustainable state*.

Let us define γ in the space $\Pi = \prod_{t=0}^{\infty} \Delta$, as

$$\gamma(\{\mathbb{X}(t)\}) = \sum_{t\in\mathbb{N}} \delta_t.$$

Observe that $\gamma : \Pi \to [0, \infty]$. Let $\Pi(\mathbb{X}_0) \subset \Pi$ be the closed, convex set of programs from $\mathbb{X}_0$. A minimal value loss program from $\mathbb{X}_0$ can be characterized as the solution of the following optimization problem, $P(\mathbb{X}_0)$.

Proposition 2 [7] *γ is convex, lower semi continuous in the product topology. There exists a good program $\{\mathbb{X}^*(t)\}$ minimizer of*

$$P(\mathbb{X}_0) \quad \begin{cases} min\ \gamma(\{\mathbb{X}(t)\}) \\ s.t.\ \ \{\mathbb{X}(t)\} \in \Pi(\mathbb{X}_0) \end{cases}$$

Proof Let $\gamma_T(\{\mathbb{X}(t)\}_{t=0}^T) = \sum_{t=0}^{T-1} \delta_t$. For every T, γ_T is convex and continuous in the product topology and $\gamma_T \leq \gamma_{T+1}$. Hence γ is the increasing limit of continuous convex functions, it is therefore convex, lower semi continuous. We know there is at least one good program, so $\min \gamma(\cdot) < \infty$. Then there is a minimizer $\{\mathbb{X}^*(t)\}$ on $\Pi(\mathbb{X}_0)$ which is non-empty, convex and compact in the product topology such that $0 \leq \gamma(\{\mathbb{X}^*(t)\}) < \infty$. □

Before presenting the equivalence result announced above, we present a technical lemma

Lemma 1 *Any maximal program is good.*

Proof Let $\{\mathbb{X}^*(t)\}$ be a maximal program from $\mathbb{X}_0$ and $\{\mathbb{X}(t)\}$ be *any* good program from $\mathbb{X}_0$. We know that

$$\sum_{t=0}^{T-1} \sum_{i \in I} [U_i(u_i(t)) - U_i(u_i^*(t))] = J_T - J_T^* \qquad \text{for all } T \tag{12}$$

where J_T^* stands for $J_T(\{\mathbb{X}^*(t)\}_{t=0}^T)$. As $\{\mathbb{X}^*(t)\}$ is maximal,

$$0 \ \geq \ \liminf_T \sum_{t=0}^{T} \sum_{i \in I} [U_i(u_i(t)) - U_i(u_i^*(t))] = J - \lim_T J_T^*$$

J is a finite real number, and so $\{\mathbb{X}^*(t)\}$ cannot be bad and the proposition follows. □

We are now in condition of showing that

Theorem 3 *Let $\{\mathbb{X}(t)\}$ be a program from $\mathbb{X}_0$. Then the following are equivalent:* (i) $\{\mathbb{X}(t)\}$ *is optimal,* (ii) $\{\mathbb{X}(t)\}$ *is maximal,* (iii) $\gamma(\{\mathbb{X}(t)\}) = \min P(\mathbb{X}_0)$.

[7] The proof we are presenting is an adaptation of Dana et al. (2006, Proposition 1.4.2). Being a very brief one, we include it for the sake of completeness.

Proof (i) $\Leftrightarrow$ (ii): The implication (i) $\Rightarrow$ (ii) is trivial from the definitions. To see (ii) $\Rightarrow$ (i), let $\{\mathbb{X}(t)^*\}$ be a maximal program. The proposition above assures that it is good, then letting $T \to \infty$ in the rhs of (12) we get

$$\limsup_T \sum_{t=0}^{T} \sum_{i \in I} [U_i(u_i(t)) - U_i(u_i^*(t))] = \left(\lim_T J_T \right) - J^*.$$

As the limit in the rhs above is defined the sought implication follows.

Before proving the equivalence between (i) and (iii), we develop a useful equality valid when considering two good programs from the same initial state. Let $\{\mathbb{X}(t)\}$ and $\{\mathbb{X}^*(t)\}$ be such programs, by (9)

$$\begin{aligned} &\lim_T \sum_{t=0}^{T-1} \sum_{i \in I} [U_i(u_i(t)) - U_i(u_i^*(t))] = \lim_T [J_T - J_T^*] \\ &= -\lim_T \sum_{t=0}^{T} \delta_t + \lim_T \sum_{t=0}^{T} \delta_t^* \\ &\quad + \lim_T r \sum_{i \in I} \left[n_i(\bar{x}_i^*(t) - \bar{x}_i(t)) + \sum_{j=2}^{n_i} (j-1)(x_i^{*j}(T) - x_i^j(T)) \right] \\ &\Rightarrow \sum_{t=0}^{\infty} \sum_{i \in I} [U_i(u_i(t)) - U_i(u_i^*(t))] = -\sum_{t=0}^{\infty} \delta_t + \sum_{t=0}^{\infty} \delta_t^* = -\gamma + \gamma^* \end{aligned} \tag{13}$$

where the last equation comes from $\lim_t \mathbb{X}^*(t) = \lim_t \mathbb{X}(t) = \hat{\mathbb{X}}$ as it was shown in Theorem 2. In the equation above and in the sequel, we use the notation $\gamma = \gamma(\{\mathbb{X}(t)\})$ and $\gamma^* = \gamma(\{\mathbb{X}^*(t)\}_{t \in \mathbb{N}})$.

(iii) $\Rightarrow$ (ii): Let $\{\mathbb{X}^*(t)\}$ be a minimizer of γ, we recall that $\{\mathbb{X}^*(t)\}$ is good. Let $\{\mathbb{X}(t)\}$ be another program from $\mathbb{X}_0$. Either it is good and by (13) we have

$$\sum_{t=0}^{\infty} \sum_{i \in I} [U_i(u_i(t)) - U_i(u_i^*(t))] = -\gamma + \gamma^* \leq 0$$

since $\{\mathbb{X}^*(t)\}$ is a minimizer of γ, or it is bad and then

$$\lim_T \sum_{t=0}^{T} \sum_{i \in I} [U_i(u_i(t)) - U_i(u_i^*(t))] = (\lim_T J_T) - J^* = -\infty.$$

Hence $\{\mathbb{X}^*(t)\}$ is optimal.

(ii) $\Rightarrow$ (iii): Conversely if $\{\mathbb{X}^*(t)\}$ is optimal, then from Lemma 1 it is good. If $\{\mathbb{X}(t)\}$ is any good program, by (13) we have

$$\gamma^* - \gamma = \lim_T \sum_{t=0}^{T} \sum_{i \in I} [U_i(u_i(t)) - U_i(u_i^*(t))] \leq 0$$

since $\{\mathbb{X}^*(t)\}$ is optimal. If $\{\mathbb{X}(t)\}$ is a bad program then $\gamma(\{\mathbb{X}(t)\}) = +\infty$. It follows that $\{\mathbb{X}^*(t)\}$ is a minimizer of γ. □

Directly as a consequence of the theorem above and Proposition 2 we have the following,

Theorem 4 *There exists an optimal program from any* $\mathbb{X}_0$.

We are now in condition of proving the existence of a sustainable state, defined as

Definition 7 A state $\mathbb{X} \in \Delta$ is called *sustainable* if it is invariant under an optimal program.

Corollary 2 *The golden rule stock* $\hat{\mathbb{X}}$ *is sustainable, i.e., if* $\mathbb{X}_0 = \hat{\mathbb{X}}$ *then the program* $\mathbb{X}^*(t) = \hat{\mathbb{X}}$ *for all* t *is optimal. Furthermore,* $\hat{\mathbb{X}}$ *is the unique sustainable state.*

Proof From the definition of $\delta(\cdot, \cdot)$, we know that $\delta(\hat{\mathbb{X}}, \hat{\mathbb{X}}) = 0$ and obviously the program $\mathbb{X}(t) = \hat{\mathbb{X}}$ for all t is a minimizer of γ.

Let us consider a state $\tilde{\mathbb{X}} \neq \hat{\mathbb{X}}$ such that $(\tilde{\mathbb{X}}, \tilde{\mathbb{X}}) \in \Delta$. By Remark 2 we have

$$\delta(\tilde{\mathbb{X}}, \tilde{\mathbb{X}}) = \sum_{i \in I} [U_i(\hat{u}_i) - U_i(\tilde{x}_i)] \geq 0$$

and due to the strict concavity of U_i we know that $\delta(\tilde{\mathbb{X}}, \tilde{\mathbb{X}}) > 0$ unless $\hat{u}_i = \tilde{x}_i$ for all $i \in I$. Thus, the stationary program $\mathbb{X}(t) = \tilde{\mathbb{X}} \neq \hat{\mathbb{X}}$ for all t is a bad program that cannot be optimal. □

In a recent paper, Zaslavski exercises Occam's razor and proves the existence of optimal programs for a general model, making no assumptions on the set Ω or the concavity of the felicity function (Zaslavski 2007). He proves that there is an optimal program from $\mathbb{X}_0$ if there is a good program from $\mathbb{X}_0$ and every good program converges to a state $\bar{\mathbb{X}}$. This state $\bar{\mathbb{X}}$ must fulfill some particular properties including that $(\bar{\mathbb{X}}, \bar{\mathbb{X}})$ is an interior point of the transition possibility set. In our particular case, there is at least one good program from *any* initial state and every good program converges to the golden rule stock, $\hat{\mathbb{X}}$ (see Theorem 2 and the paragraph preceding it). But, $(\hat{\mathbb{X}}, \hat{\mathbb{X}})$ does not belong to the interior of Ω. The proof presented here is an alternative to Zaslavski's proof, valid for our model, that is also heavily based on the existence of a good program from every initial state and the convergence of good programs and circumvents the interiority of $(\hat{\mathbb{X}}, \hat{\mathbb{X}})$.

6 Value function and preorder in Δ

In this section we define a value function for our model that allows the development of a theory of undiscounted dynamic programming. We also introduce a preorder in Δ that provides an alternative characterization of the golden rule stock and the value loss function. This preorder is not necessary in the proofs of the main results of this work, however, it is presented here as it may have independent interest.

Let us define $V : \Delta \to \mathbb{R}$, the value function,[8] as

$$P(\mathbb{X}) \begin{cases} V(\mathbb{X}) = \max J(\{\mathbb{X}(t)\}) \\ \qquad\qquad \text{s.t. } \{\mathbb{X}(t)\} \text{ is a program from } \mathbb{X} \end{cases}$$

Thanks to (13) and Theorem 3, we know that there is a solution to $P(\mathbb{X})$ such that $V(\mathbb{X}) > -\infty$. Furthermore, $V(\mathbb{X}) \le M$ for all $\mathbb{X} \in \Delta$ due to Proposition 1.

Let $F(\mathbb{X}) \subseteq (\ell_+^\infty)^k$ be the set of all harvesting sequences obtained with a feasible program from $\mathbb{X}$,

$$F(\mathbb{X}) = \left\{ \{\boldsymbol{u}(t)\}_{t\in\mathbb{N}} \Big/ \exists \{\mathbb{X}(t)\} \text{ with } \mathbb{X}(0) = \mathbb{X} \text{ and } (\mathbb{X}(t), \mathbb{X}(t+1)) \in \Omega_{\boldsymbol{u}(t)} \ \forall t \right\}.$$

Observe that $F(\mathbb{X}) \subseteq F(\mathbb{Y})$ implies $V_T(\mathbb{X}) \le V_T(\mathbb{Y})$ for all T where [9]

$$\begin{cases} V_T(\mathbb{X}) = \max \ J_T(\{\mathbb{X}(t)\}_{t=0}^T) \\ \qquad\qquad \text{s.t. } (\mathbb{X}(t), \mathbb{X}(t+1)) \in \Omega \qquad t = 0, \ldots, T-1 \\ \qquad\qquad\quad \text{with } \mathbb{X}(0) = \mathbb{X} \text{ given.} \end{cases}$$

Lemma 2 *The following equivalence holds*

$$F(\mathbb{X}) \subseteq F(\mathbb{Y}) \iff \bar{x}_i + \sum_{j=0}^{t} x_i^{n_i-j} \le \bar{y}_i + \sum_{j=0}^{t} y_i^{n_i-j} \quad forall$$
$$i \in I, \quad t = 0, \ldots, n_i - 1. \tag{14}$$

The lemma follows easily from the definition of $F(\mathbb{X})$.

Definition 8 Given $\mathbb{X}$ and $\mathbb{Y} \in \Delta$ we say that $\mathbb{X} \preceq \mathbb{Y}$ if $F(\mathbb{X}) \subseteq F(\mathbb{Y})$.

Notice that the relationship $\preceq$ defined in Δ is a preorder as it is reflexive and transitive. However it is not antisymmetric, because $F(\mathbb{X}) \subseteq F(\mathbb{Y})$ and $F(\mathbb{Y}) \subseteq F(\mathbb{X})$ would imply $\bar{x}_i + x_i^{n_i} = \bar{y}_i + y_i^{n_i}$ and $x_i^j = y_i^j$ for all $j = 1, \ldots, n_i - 1$ but not $\mathbb{X} = \mathbb{Y}$.

[8] This value function was introduced in Brock and Majumdar (1988) and later used by Khan and Mitra (2006). In Zaslavski (2007) it is used with the opposite sign.

[9] However, $V_T(\mathbb{X}) \le V_T(\mathbb{Y}) \not\Rightarrow F(\mathbb{X}) \subseteq F(\mathbb{Y})$.

When restricted to $\Delta^0 = \{\mathbb{X} \in \Delta / \bar{x}_i = 0 \text{ for all } i \in I\}$ this ambiguity is eliminated and $\preceq$ turns into a partial order.[10]

Once we endow the set Δ with the pre-order $\preceq$, we can give an alternative characterization of the golden rule stock and re-establish the validity of the value-loss function.

Proposition 3 *The golden rule stock, $\hat{\mathbb{X}}$, is the unique solution in Δ^0, to the following problem* [11]

$$(P) \qquad \max_{\mathbb{X}} \left\{ \sum_{i \in I} U_i(u_i) \Big/ \exists\, \mathbb{Y} \textit{ such that } (\mathbb{X}, \mathbb{Y}) \in \Omega_{\boldsymbol{u}}\,,\ \mathbb{X} \preceq \mathbb{Y} \right\}$$

Proof Recall that $(\mathbb{X}, \mathbb{Y}) \in \Omega_{\boldsymbol{u}}$ is equivalent to $u_i = \bar{x}_i + x_i^{n_i} - \bar{y}_i \geq 0$ and $y_i^j = x_i^{j-1}$ for all $j = 2, \ldots, n_i$. Using this last set of equations, condition (14) can be expressed as

$$\begin{aligned}
\bar{y}_i + x_i^{n_i-1} &\geq \bar{x}_i + x_i^{n_i} \\
\bar{y}_i + x_i^{n_i-1} + x_i^{n_i-2} &\geq \bar{x}_i + x_i^{n_i} + x_i^{n_i-1} \\
&\vdots \\
\bar{y}_i + x_i^{n_i-1} + x_i^{n_i-2} + \cdots + x_i^1 &\geq \bar{x}_i + x_i^{n_i} + x_i^{n_i-1} + \cdots + x_i^2 \\
\bar{y}_i + x_i^{n_i-1} + x_i^{n_i-2} + \cdots + x_i^1 + y_i^1 &\geq \bar{x}_i + x_i^{n_i} + x_i^{n_i-1} + \cdots + x_i^2 + x_i^1
\end{aligned}$$

and then reduced to

$$\begin{aligned}
&x_i^j \geq \bar{x}_i + x_i^{n_i} - \bar{y}_i = u_i && \forall i \in I,\ j = 1, \ldots, n_i - 1. \\
&y_i^1 \geq u^i && \text{(In fact, } y_i^1 = u_i \text{ due to (3))}
\end{aligned}$$

Hence the optimization problem can be stated as

$$(P) \qquad \max_{\mathbb{X} \in \Delta} \max_{\boldsymbol{u}} \left\{ \sum_{i \in I} U_i(u_i) \Big/ \bar{x}_i + x_i^{n_i} \geq u_i,\ x_i^j \geq u_i \geq 0 \quad \forall i \in I,\ j = 1, \ldots, n_i - 1 \right\}$$

[10] The set Δ^0 in introduced in Cominetti and Piazza (2009). It plays an important role in the discounted setting, where sufficient conditions involving the discount factor can be found to assure that optimal programs reach Δ^0 in a finite number of steps and remain there afterwards, i.e., no over-mature trees are allowed along optimal programs after some steps.

[11] Compare with Definition 2. The new definition is formally identical to the one provided by Gale (1967).

In order to solve this problem we introduce the Lagrangian

$$\mathcal{L} = \sum_{i\in I}\left[U_i(u_i) + \mu_i u_i + \nu_i^{n_i}(\bar{x}_i + x_i^{n_i} - u_i) + \sum_{j=1}^{n_i-1}\nu_i^j(x_i^j - u_i)\right]$$
$$+\theta\left[S - \sum_{i\in I}(\bar{x}_i + \sum_{j=1}^{n_i} x_i^j)\right].$$

We claim that the point $\bar{x}_i = 0$, $u_i = x_i^j$ and $U_i'(u_i)/n_i = r$ if $u_i > 0$ and $U_i'(u_i)/n_i \leq r$ if $u_i = 0$ together with the following set of nonnegative multipliers

$$\begin{cases} \theta = \nu_i^j = r = U_1'(u_1)/n_1 \\ \mu_i = n_i\left[r - U_i'(u_i)/n_i\right] \geq 0 \end{cases}$$

satisfy the Karush-Kuhn-Tucker conditions and thus the proposed point is a solution to the concave problem. Furthermore, the solution characterized is precisely $\hat{\mathbb{X}}$. We leave the details to the reader. □

Observe that the condition $\hat{\mathbb{X}} \in \Delta^0$ is necessary to assure uniqueness of the solution. It is easy to see that without this additional condition, the solution set of the optimization problem is the segment

$$S(P) = \left\{\mathbb{X} \in \Delta \big/ x_i^j = \hat{x}_i,\ j=1,\ldots,n_i-1 \text{ and } \bar{x}_i + x_i^{n_i} = \hat{x}_i\right\}$$

Using that the solution point is a saddle point of the Lagrangian, we can deduce

$$\sum_{i\in I} U_i(\hat{u}_i) \geq \sum_{i\in I}\left\{U_i(u_i) + n_i[r - U_i'(\hat{u}_i)/n_i]\,u_i + r(\bar{x}_i + x_i^{n_i} - u_i)\right.$$
$$\left.+\sum_{j=1}^{n_i-1} r(x_i^j - u_i)\right\}$$
$$= \sum_{i\in I}\left\{U_i(u_i) + n_i[r - U_i'(\hat{u}_i)/n_i]\,u_i\right\} + rS - \sum_{i\in I} rn_iu_i$$
$$= \sum_{i\in I} U_i(u_i) - U_i(\hat{u}_i)\,u_i + rS \geq \sum_{i\in I} U_i(u_i) - rn_iu_i + rS$$

which is exactly (5) and the non-negativity of the value loss function follows directly.

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