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Adriana Piazza & Bernardo K. Pagnoncelli

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# The optimal harvesting problem under price uncertainty

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**Abstract** In this paper we study the exploitation of a one species forest plantation when timber price is governed by a stochastic process. The work focuses on providing closed expressions for the optimal harvesting policy in terms of the parameters of the price process and the discount factor, with finite and infinite time horizon. We assume that harvest is restricted to mature trees older than a certain age and that growth and natural mortality after maturity are neglected. We use stochastic dynamic programming techniques to characterize the optimal policy and we model price using a geometric Brownian motion and an Ornstein–Uhlenbeck process. In the first case we completely characterize the optimal policy for all possible choices of the parameters. In the second case we provide sufficient conditions, based on explicit expressions for reservation prices, assuring that harvesting everything available is optimal. In addition, for the Ornstein–Uhlenbeck case we propose a policy based on a reservation price that performs well in numerical simulations. In both cases we solve the problem for *every* initial condition and the best policy is obtained endogenously, that is, without imposing any ad hoc restrictions such as maximum sustained yield or convergence to a predefined final state.

**Keywords** Stochastic dynamic programming · Forest management · Optimal harvesting · Price uncertainty

## 1 Introduction

The exploitation of forest resources for economic benefits dates back many centuries. In his seminal paper of 1849, Faustmann proposed the correct formula to valuate a standing forest

A. Piazza (✉)
Departamento de Matemática, Universidad Técnica Federico Santa María, Avenida España 1680, Casilla 110-V, Valparaíso, Chile
e-mail: adriana.piazza@usm.cl

B. K. Pagnoncelli
Escuela de Negocios, Universidad Adolfo Ibáñez, Diagonal las Torres 2640, Peñalolén, Santiago, Chile
e-mail: bernardo.pagnoncelli@uai.cl

plantation and to determine its optimal rotation length (see a translation of the original German article in Faustmann 1995). The simplicity of Faustmann result comes from the fact that all the trees are considered identical and prices are assumed to be known and constant over time. A wide literature has developed since then, in numerous directions that go from dealing with more sophisticated growth models to considering non-timber forest products such as tourism and environmental services or introducing uncertainty in forest growth, prices and costs. In this work, we generalize Faustmann work by considering stochastic prices at the same time that we move from a stand-level to a forest-level model, allowing for trees of different ages. Even more, we aim to determine theoretically the optimal harvesting policy without imposing any ad hoc restrictions.

The generalization of Faustmann problem to one where different age classes are allowed was already known before Faustmann' times but its complete resolution remains open even today. The first approaches to find an age-specific optimal harvesting policy comprised the additional assumption that the forest converged to a predefined final state (typically, the normal forest) or the imposition of constraints on timber flow (see the survey Tahvonen 2004 and references therein). Later on, linear programming was applied to the numerical resolution of the optimal harvesting problem (see, for example, Davis and Johnson 1987; Leuschner 1990).

We consider the forest as a collection of multiple stands of even aged trees and, following the model paradigm used by Mitra and Wan (1985), we do not consider the number of trees in each age class to describe the structure of the forest; instead, we consider the area occupied by the corresponding age class. This is valid for managed planted forests, since trees are planted within a predefined and constant distance of each other. Mitra and Wan (1985) find the optimal rotation period of a single age class forest, as a function of the biomass coefficients and the discount factor affecting future benefits. They also find examples where the optimal solution is a periodic cycle and does not converge to the normal forest. The issue was further investigated by Salo and Tahvonen (2003) showing that every initial condition close enough to the normal forest yields a periodic optimal trajectory, so that this state is not even a local attractor. They conjectured that the optimal trajectory starting from any initial state would became periodic in the long-run, proving it for the case of a two-stand forest (Salo and Tahvonen 2002).

The model we use was proposed by Rapaport et al. (2003) and is a simplification of the Mitra and Wan model. It represents a forest plantation where harvest is forbidden before a pre-defined maturity age and growth after maturity is neglected as well as natural mortality. We consider the maturity age $n$, as exogenously fixed. A natural choice for $n$ could be the optimal rotation period characterized by Mitra and Wan (1985).

In Rapaport et al. (2003), the authors define a *greedy policy* as one in which every tree is harvested as soon as it reaches maturity. They show that, in the deterministic setting with linear timber price, every optimal trajectory is greedy and the optimization problem can be trivially solved. In a more general case, when the benefit obtained from the timber is represented by a concave function of the harvested timber volume, the authors prove that every optimal trajectory becomes greedy in, at most, twice the maturity age. This implies that the infinite time horizon problem can be stated as a finite dimensional optimization problem and can be numerically solved.

An extension of the Rapaport et al. model to a mixed forest composed by several species of different maturity ages is considered in Cominetti and Piazza (2009). The authors prove the existence and uniqueness of a steady state and show that an optimally managed forest converges towards the steady state under a mild additional condition on the maturity ages, with *no* restriction on the discount factor.[1]

Both works Rapaport et al. (2003) and Cominetti and Piazza (2009) describe the structure and behavior of the optimal policy but ignore any source of uncertainty. In real-life timber prices are uncertain and the need for the inclusion of random price process has been highlighted by several authors (see, for example, Alonso-Ayuso et al. 2011; Clarke and Reed 1989; Lohmander 2000; McGough et al. 2004). Financial aspects of the problem are studied in Mosquera et al. (2011), where the authors study insurance contracts for forestry firms that want to be protected against price volatility.

In the static setting, Reeves and Haight (2000) proposed a mean variance optimization model and estimated returns using time series on historical data. In the dynamic setting, the question of which process best represents timber prices is still the object of discussion inside the resource economists community. Some authors assume normality with respect to price fluctuation and suggest that a geometric Brownian motion (GBM) should be used as it is consistent with an informationally efficient market (Brazee and Mendelsohn 1988; Clarke and Reed 1989; Thomson 1992). However, other authors (Alvarez and Koskela 2005; Gjolberg and Guttormsen 2002) point out that a mean reverting process, or Ornstein–Uhlenbeck (O–U) is a better description of the timber price path due to empirical data that has been collected for several species. It is not our intention to go any further into this discussion, we refer the reader to Insley and Rollins (2005) and references therein. As the issue seems far from being settled, we decided to consider both GBM and O–U in this paper.

Most theoretical work on price uncertainty in forestry has been conducted with stand-level models and, to our knowledge, our work is one of the few to address optimal harvesting at the forest level with multiple age classes and price uncertainty. In Tahvonen and Kallio (2006), the authors discuss this problem and conduct numerical experiments to study how the optimal harvesting policy changes with the inclusion of stochastic prices and risk aversion.

In our paper we consider the same prices processes as in Tahvonen and Kallio (2006) and work with a simpler model, but we characterize the optimal policy based on closed form expressions. When the price is governed by a GBM, we completely characterize the optimal harvesting policy as a function of the discount factor and the drift of the price process. When the prices are modeled as an O–U, we find a threshold (*reservation price*) such that if the current price exceeds it we can assure that it is optimal to harvest everything available, regardless of the future. In our paper we restrict ourselves to the risk neutral case, but we believe the results presented here can be extended to the risk averse case.

It is interesting to note that, although the problem we consider has similarities with multi-period portfolio selection problems, the age structure of the forest and the fact that the harvesting decisions ultimately determine how much timber will be available in the future make it impossible to apply the results developed for the portfolio problem. For example, in Blomvall and Shapiro (2006) the authors use similar techniques to characterize the optimal policy in a multistage stochastic portfolio selection problem under logarithmic and exponential utility. In their model assets can be bought and sold at anytime and in any quantity, which is a reasonable assumption for such problems. Such assumption does not

[1] The fact that global convergence is obtained in a discounted utility framework with no restriction on the discount factor is a distinguishing feature of this work as most convergence theorems are either local or assume discount factors close to one.

make sense for harvesting problems since one can only harvest what is available at a given time period.

The paper is organized as follows: Section 2 presents the forest growth model and the price processes we are considering as well as the optimization problem to be solved. Section 3 is the main contribution of this work and starts with the derivation of the stochastic dynamic programming equations that are the main tools to tackle the optimal harvesting problem. It also comprises the derivation of our theoretical results characterizing the optimal harvesting policies and some numerical experiments for self-generated instances. Finally, Sect. 4 concludes this work and brings up some future lines of research. All the proofs are relegated to the Appendix.

## 2 Model formulation

Let us consider a forest of total area $S$ occupied by a one species forest with maturity age of $n$ years. In contrast with the case of wild forests, the state of a forest plantation may be described by specifying the areas occupied by trees of different ages, making the assumption that tress are planted within a pre-specified and constant distance of each other.

For each period $t \in \mathbb{N}$ we denote $x_{a,t} \geq 0$ the area of trees of age $a = 1, \ldots, n$ in year $t$, and $\bar{x}_t \geq 0$ the area occupied by trees beyond maturity (older than $n$). Using a single state variable to represent the over-mature trees conveys the underlying assumption that the growth and the natural mortality of trees are negligible beyond maturity. Each period we must decide how much land $c_t \geq 0$ to harvest. Assuming that only mature trees can be harvested, we must have

$$0 \leq c_t \leq \bar{x}_t + x_{n,t}, \tag{1}$$

and then the area not harvested in that period will comprise the over-mature trees at the next step, namely

$$\bar{x}_{t+1} = \bar{x}_t + x_{n,t} - c_t. \tag{2}$$

We neglect natural mortality at every age, again an assumption valid in managed forest plantations but not in wild forests. Hence, the transition between age classes is given by

$$x_{a+1,t+1} = x_{a,t} \quad \forall a = 1, \ldots, n-1. \tag{3}$$

The harvested area is immediately allocated to new seedlings that will comprise the 1 year old trees in the following year:

$$x_{1,t+1} = c_t. \tag{4}$$

We represent the *state* of the tree population by the vector state

$$\mathbb{X} = \begin{pmatrix} \bar{x} \\ x_n \\ \vdots \\ x_1 \end{pmatrix}$$

and the dynamics described by Eqs. (2), (3) and (4) can be represented as follows:

$$\mathbb{X}_{t+1} = A\mathbb{X}_t + Bc_t, \tag{5}$$

where

$$A = \begin{pmatrix} 1 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The expression of constraint (1) in terms of the defined matrices is

$$0 \le c_t \le CA\mathbb{X}_t \quad \text{where} \quad C = (10.\dots..0). \tag{6}$$

The order of events is the following: at every period $t$ the decision maker observes the state of the forest and the current price and decides how much she will harvest, $c_t$, obtaining a benefit $p_t c_t$. Her decision changes the current state of the system from $\mathbb{X}_t$ to $\mathbb{X}_{t+1}$ according to (5). Then timber price $p_{t+1}$ is observed, leaving her in position of deciding the following harvest $c_{t+1}$. The sequence of events has the form

$$(p_1, \mathbb{X}_1) \rightsquigarrow c_1 \rightsquigarrow \mathbb{X}_2 \rightsquigarrow p_2 \rightsquigarrow c_2 \rightsquigarrow \mathbb{X}_3 \rightsquigarrow \cdots$$

We will study two different dynamics of prices: a geometric Brownian motion (GBM) and a mean reverting arithmetic Ornstein-Uhlenbeck (O-U) process. The first choice is justified by its extensive use to model asset prices in financial markets and, therefore, represents a natural choice for timber prices. Nevertheless, the GBM does not capture some behaviors of price movement such as mean reversion, which is best emulated by the O–U process. Since both processes are well-known, we define them without further detail.

$$dp_t = \mu p_t dt + \sigma p_t dW_t \qquad \text{(GBM)}, \tag{7}$$

$$dp_t = \eta(\bar{p} - p_t)dt + \sigma dW_t \qquad \text{(O–U)}, \tag{8}$$

where $\mu \in \mathbb{R}$ is the *drift* of the GBM, $\sigma > 0$ is the constant variance, $\eta > 0$ is the mean reversion rate of the O–U to an equilibrium $\bar{p}$, and $W_t$ denotes the Wiener process.

We will often compute the conditional expectation of both processes, which can be explicitly calculated since closed-form solutions are known (see Dixit and Pindyck 1994 for details):

$$\mathbb{E}_{|p_t}[p_{t+1}] = p_t e^{\mu}, \quad \text{for all } t, \qquad \text{(GBM)}, \tag{9}$$

$$\mathbb{E}_{|p_t}[p_{t+1}] = p_t e^{-\eta} + \bar{p}(1 - e^{-\eta}), \quad \text{for all } t, \qquad \text{(O–U)}, \tag{10}$$

where $\mathbb{E}_{|\cdot}[\cdot]$ denotes the conditional expectation of a random variable.

The objective function is defined by the expected sum of the instantaneous benefits $(p_t c_t)$ discounted by the factor $\delta \in (0, 1)$,

$$\mathbb{E}_{|p_1}\left[\sum_{t \in \mathcal{T}} \delta^{t-1} p_t c_t\right], \tag{11}$$

where we consider two different types of sets $\mathcal{T}$: (i) $\mathcal{T} = \{1, \dots, T\}$ for the finite horizon case, where $T$ is the time horizon of the problem, and, (ii) $\mathcal{T} = \mathbb{N}$ for the infinite horizon case. The control $c_t$ in each period will depend only on the current state of the forest and price through the *decision function* $c_t = \pi_t(\mathbb{X}_t, p_t)$. A sequence $\Pi = \{\pi_t\}_{t \in \mathcal{T}}$ is called a *policy*. Of course, a policy is *feasible* if $\mathbb{X}_t$ and $c_t = \pi_t(\mathbb{X}_t, p_t)$ satisfy (5) and (6) for every possible value of $\mathbb{X}_t$ and $p_t$ at every instant $t \in \mathcal{T}$. Observe that the non-negativity of the state variables $\bar{x}_t$ and $x_{a,t}$ for $a = 1, \dots, n$ is assured by (5) and (6).

The expected benefit of a policy $\Pi$ given an initial state $\mathbb{X}_1$ and an initial price $p_1$ is

$$Q_1^{\Pi}(\mathbb{X}_1, p_1) = \mathbb{E}_{|p_1}\left[\sum_{t \in \mathcal{T}} \delta^{t-1} p_t \pi_t(\mathbb{X}_t, p_t)\right]. \tag{12}$$

The problem is then to find a feasible policy that maximizes (12),

$$V_1(\mathbb{X}_1, p_1) = \begin{cases} \text{Max}_{\Pi} & Q_1^{\Pi}(\mathbb{X}_1, p_1) \\ \text{s.t.} & \Pi \text{ is a feasible policy.} \end{cases} \tag{13}$$

In the sequel, we will also use the expected discounted benefit from an intermediate step

$$Q_t^{\Pi}(\mathbb{X}_t, p_t) = \mathbb{E}_{|p_t}\left[\sum_{s \in \mathcal{T}_t} \delta^{s-t} p_s \pi_s(\mathbb{X}_s, p_s)\right],$$

where $\mathcal{T}_t = \{s \in \mathcal{T} : s \geq t\}$, and the corresponding value function

$$V_t(\mathbb{X}_t, p_t) = \begin{cases} \text{Max}_{\Pi} & Q_t^{\Pi}(p_t, \mathbb{X}_t) \\ \text{s.t.} & \Pi \text{ is a feasible policy.} \end{cases}$$

## 3 Finding optimal policies

Let us start this section by deriving the stochastic dynamic programming equations for our problem. From the definitions of $V_1$ and $Q_1^{\Pi}$, it is straightforward to see that:

$$V_1(\mathbb{X}_1, p_1) = \underset{\Pi}{\text{Max}}\, Q_1^{\Pi}(\mathbb{X}_1, p_1) = \underset{\Pi}{\text{Max}}\, \mathbb{E}_{|p_1}\left[p_1 c_1 + \delta \sum_{t \in \mathcal{T}_2} \delta^{t-2} p_t c_t\right], \tag{14}$$

where $c_t$ stands for $\pi_t(\mathbb{X}_t, p_t)$. Formulation (14) is an example of a multi-stage stochastic programming problem (see for example Chapter 3 of Shapiro et al. 2009). Following Shapiro (2011), we can write a nested formulation of (14) as follows[2]:

$$V_1(\mathbb{X}_1, p_1) = \underset{c_1}{\text{Max}}\, c_1 p_1 + \delta \mathbb{E}_{|p_1}\left[\underset{c_2}{\text{Max}}\, p_2 c_2 + \delta \mathbb{E}_{|p_2}\left[\underset{c_3}{\text{Max}}\, p_3 c_3 \cdots\right]\right],$$

where the expected values are not conditional on the entire history of the process because the processes we consider in this paper are Markovian. We can write explicitly the dynamic programming equations for $t \in \mathcal{T}$ in the following way:

$$\begin{aligned} V_t(\mathbb{X}_t, p_t) = \quad & \underset{c_t}{\text{Max}}\, p_t c_t + \delta \mathbb{E}_{|p_t}[V_{t+1}(A\mathbb{X}_t + Bc_t, p_{t+1})] \\ \text{s.t.} \quad & 0 \leq c_t \leq CA\mathbb{X}_t, \end{aligned} \tag{15}$$

where by definition $V_{T+1} \equiv 0$, in the finite horizon case.

### 3.1 Optimal policies for GBM

A well studied and important policy in forestry is the so-called *greedy policy* (GP), which consists in harvesting everything available at any time, that is

[2] This derivation is only valid if (11) converges.

$$c_t = CA\mathbb{X}_t \quad \forall t \in \mathcal{T}.$$

Before showing rigorously which is the optimal policy, we propose an intuitive rule of thumb to decide whether the GP is optimal or not. It is natural to think that if discounted expected prices in the future are smaller than the present price, the decision maker should harvest everything available since on average discounted prices will drop. If the price process is a GBM, and assuming that the *future is just one period ahead*, this condition can be translated by

$$\begin{aligned} \delta\mathbb{E}_{|p_t}[p_{t+1}] &< p_t, \\ \delta p_t e^{\mu} &< p_t, \end{aligned}$$

which is equivalent to

$$\delta e^{\mu} < 1. \tag{16}$$

The economic intuition for the GBM case is that if condition (16) is satisfied then the discounted expected future price of time $t + 1$ is smaller than the present price $p_t$ for all $t$ and therefore the optimal policy is to harvest everything available. The reader has no reason to believe that if condition (16) is satisfied then the optimal policy is to harvest everything available. First because we are only considering expected prices. Secondly, and most important, because we are only looking one period ahead: it could be the case that after two or more periods prices would rise and so harvesting everything available now is not the best option.

### 3.1.1 When is the greedy policy optimal?

Surprisingly, the intuitive condition (16) derived with a very simple reasoning is exactly what we need to formalize the optimality of the GP. Before stating the theorem, we need a brief technical lemma showing that the value function is well defined in the infinite horizon case.

**Lemma 1** *If Condition (16) holds, the value function $V_t(\cdot,\cdot)$ is well defined.*

**Theorem 1** *Consider problem (13) and assume prices evolve according to (7). If condition (16) holds, the GP is optimal.*[3]

All the proofs are presented in the Appendix.

### 3.1.2 Managerial insights

An immediate consequence of condition (16) is that if the drift $\mu$ is less than or equal to zero then the condition is satisfied for all $\delta$. Therefore, according to Theorem 1, it is better to harvest as soon as timber becomes available. The intuition behind this result is that, because of (9), $\mu \leq 0$ implies that the expected value of future prices is less than or equal to the present price. Furthermore, the discount factor $\delta$ penalizes future benefits, making immediate harvest the best option.

[3] When Condition (16) holds with equality, the result is still valid in the finite horizon case, but, $V_t$ is not defined when $\mathcal{T} = \mathbb{N}$. To simplify the statement of the Theorem, we state the result with the strict inequality (so that it is valid in both cases), although, a slightly stronger result can be obtained with a straightforward modification of the proof.

When the drift is positive, the expected value of future prices is greater than the current observed price. Despite the optimistic forecast for future prices, condition (16) might still be satisfied for a sufficiently small $\delta$. In Fig. 1, the shaded area shows the values of $\delta$ and positive values of $\mu$ for which the sufficient condition holds.

If $\delta = 1$, then future is not discounted. In this case, if $\mu > 0$ it makes sense to wait for prices to rise and postpone the harvesting decision: since there is no discounting and prices may rise in the future, it does not make sense to harvest at the present and lose potential gains in the future. On the other hand, if $\mu \leq 0$ then it is optimal to harvest everything, even though there is no discounting, because prices will tend to fall.

A financial interpretation can be easily obtained by representing the discount factor $\delta$ by the continuously-compounded rate $e^{-r}$, where $r$ is a discount yield between zero and one that represents the interest rate. In other words, one dollar today is worth $e^r$ dollars in the next period. We have

$$\delta e^{\mu} < 1 \iff e^{-r} e^{\mu} < 1 \iff e^{\mu} < e^{r} \iff \mu < r,$$

which tells us that if the drift of the price process is not above the interest rate $r$ then harvesting everything available has greater monetary value than postponing the harvest decision and discounting according to the interest rate $r$.

We would like to close our observations highlighting the fact that condition (16) can be checked a priori, as it depends only on $\delta$ and $\mu$, which are known before any decision has to be made.

### 3.1.3 *Another optimal policy*

When condition (16) does not hold, we have that discounted expected prices grow constantly, making the value function ill-defined in the infinite horizon case. The proof of the following lemma is presented in the Appendix,

**Lemma 2** *If $\mathcal{T} = \mathbb{N}$ and Condition (16) does not hold, the value function $V_t(\cdot,\cdot)$ is always infinite.*

Hence, in this subsection we only consider the finite horizon case. Since we have that discounted expected prices are greater than the present price, it is natural to think that the decision maker should postpone the harvest as much as possible. Consequently, before the final time $T$, harvesting should be stopped altogether in order to have the maximum surface available at time $T$. However, observe that every land plot harvested and planted $n$ or more time steps before $T$, will contain mature trees available for harvesting at time $T$. Hence, it is convenient to harvest every mature tree at time $T - n$, since there is enough time for seedlings to mature before reaching $T$. Repeating this reasoning we can conjecture that the only time steps when harvesting is allowed are $T - kn$ for $k = 0, 1, \ldots, \lfloor T/n \rfloor$ and that everything available then should be harvested. This is to say,

$$c_t = \begin{cases} CA\mathbb{X}_t & \text{if } t = T - kn,\ k = 0, 1, \ldots, \lfloor T/n \rfloor, \\ 0 & \text{else.} \end{cases}$$

We call this harvesting policy, the *accumulating policy* (AP).

The proof of this intuitive claim is presented in Theorem 2,

**Theorem 2** *Consider problem (13) and assume prices evolve according to (7). If condition (16) does not hold, the accumulating policy is optimal.*

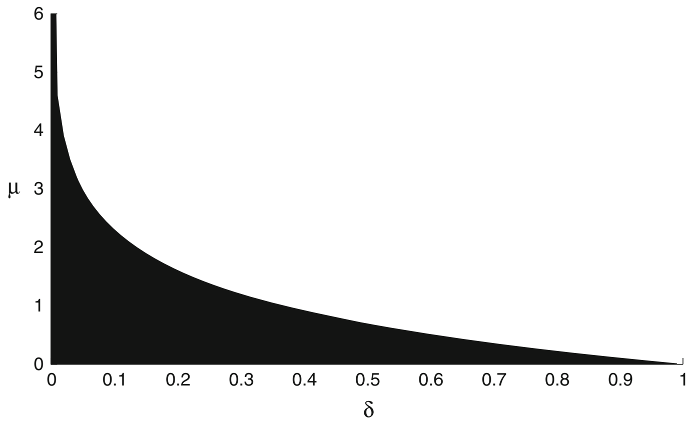


**Fig. 1** (δ, μ) values for GBM

We would like to point out that the characterization of the optimal policy in the finite horizon case is complete. Furthermore, optimal decisions for the whole interval $1, \ldots, T$ can be taken at $t = 1$ by checking if condition (16) is satisfied or not.

### *3.1.4 Numerical experiments for the finite horizon case*

In order to run the experiments, we adopted the binomial approximation developed by Cox et al. (1979) for GBM. Using the binomial tree, we found the optimal policy and verified that it agrees with Theorem 1: when the parameters δ and μ satisfy condition (16), the optimal policy found coincided with GP, whereas when condition (16) is not satisfied it coincided with AP.

To find the optimal policy we use Dynamic Programming techniques. That is to say, we start at the last period $T$ and, for every state, we calculated the optimal harvesting policy by optimizing over all possible harvesting decisions. Using (15) we moved backwards until the first period and stored the optimal policies at every time period. The value of the resulting optimal policy is simply the value function at time $t = 1$.

Table 1 shows statistics of the value of both policies for $\delta = 0.95$, $\mu = 0.1$ and $S = 1$, with initial price $p_1 = 2$ at $t = 1$. Note that in this case $\delta e^{\mu} = 1.05$ and therefore condition (16) is not satisfied. For some values of $n$ and $T$ we compute the optimal value for every possible initial state of the forest and compute the average under GP and AP. We also report the percentage by which the average AP is greater than the average GP (Avg AP/Avg GP − 1). Results are shown in rows 2, 3 and 4 of Table 1. We can see from the 4th row that the AP policy is clearly different from the GP, yielding up to 12.9% larger average percentage returns.

To further illustrate the differences between the two policies, we compare their performance when the initial state represents the youngest possible forest, that is, $\mathbb{X}_1 = (0, 0, \ldots, S) := \mathbb{X}^S$. The comparison is interesting from a managerial viewpoint: if a manager starts with a young forest she wants to know what are the implications of adopting a GP instead of the AP. We also report the percentage increment change (Avg AP/Avg GP −1) for each of the four experiments.

**Table 1** Comparison between the accumulating policy and the greedy policy

| (n, T) | (3, 10) | (4, 15) | (5, 15) | (6, 40) |
|---|---|---|---|---|
| Avg AP | 9.09 | 11.65 | 9.48 | 45.26 |
| Avg GP | 8.81 | 10.82 | 8.49 | 40.11 |
| Avg. incr. (%) | 3.2 | 7.7 | 11.7 | 12.9 |
| AP for $\mathbb{X}^S$ | 8.09 | 9.89 | 9.49 | 43.61 |
| GP for $\mathbb{X}^S$ | 7.71 | 8.54 | 9.49 | 35.89 |
| Avg. incr. for $\mathbb{X}^S$ (%) | 5.0 | 15.7 | 0 | 21.5 |

## 3.2 Optimal policies for O–U

Similarly to what we did for the GBM, we can derive intuitive conditions for the optimality of the GP:

$$\delta\mathbb{E}_{|p_t}[p_{t+1}] \le p_t,$$
$$\delta[p_t e^{-\eta} + \bar{p}(1-e^{-\eta})] \le p_t,$$

which is equivalent to

$$\frac{p_t}{\bar{p}} \ge \frac{\delta(1-e^{-\eta})}{(1-\delta e^{-\eta})}. \tag{17}$$

Note that, unlike (16) condition (17) depends on $p_t$. It remains to be seen if condition (17) can be proven to lead to greedy policies. Let the right hand side of (17) be denoted by $r$, we note that (17) suggests that if $p_t$ is greater than the threshold $r\bar{p}$ then one should harvest everything available. For the O–U case, our results are valid only for finite horizon problems. We were not able to prove that the value function is finite when the time horizon is infinite, as for the GBM case. The proof we used in the previous subsection was based in the non-negativity of $p_t$ and cannot be adapted because, following a normal distribution, $p_t$ can take negative values. In the rest of this section we only consider the finite horizon case.

The following theorem proves that if (17) holds at $t$, then it is optimal to harvest everything available at $t$, regardless of the future. We cannot affirm that the GP is optimal from $t$ on, but we assure that harvesting everything available at $t$ is optimal if (17) holds for $p_t$. The condition is sufficient but not necessary: if (17) do not hold, we are not able to discard the GP. However, we provide some numerical experiments where the threshold is used as a *reservation price*, i.e., we harvest everything available if $p_t$ is above $r\bar{p}$ and nothing if it is below. The results obtained are very satisfactory.

**Theorem 3** *Consider problem (13) and assume prices evolve according to (8). If condition (17) holds at time t, then* $c_t = CA\mathbb{X}_t$ *is optimal.*

### *3.2.1 Managerial insights*

The most important difference between (16) and (17) is that the former does not depend on $t$. While in the GBM case the decision maker can check *a priori* (at time $t = 1$) if the GP is optimal, under O-U the condition needs to be checked at every time period $t \in \mathcal{T}$.

Condition (17) imposes the same reservation price for all time periods $t$, and thus it is worth investigating the behavior of the right hand side of this expression to understand how

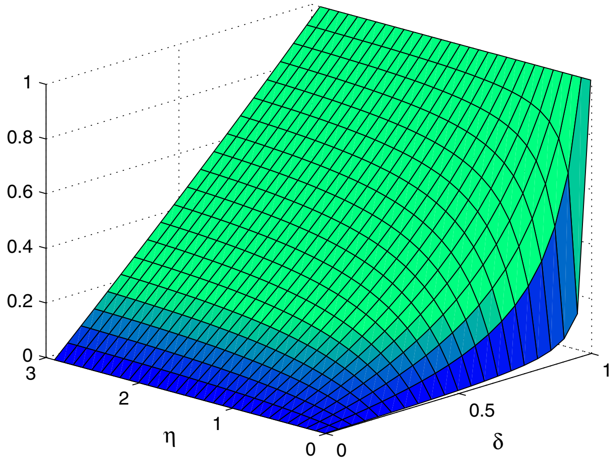


**Fig. 2** (δ, η) values for O–U

restrictive this condition really is. To this end, we study the variation of the right hand side of (17) as a function of the discount factor δ and the mean reversion rate η, that is, $r := r(\delta, \eta)$. We show in Fig. 2 the plot of function $r(\delta, \eta)$. Analytically, and also from Fig. 2, it is possible to see that $r(\delta,\eta) \leq 1$, which means, that the threshold imposed on $p_t$ by condition (17) is smaller than the equilibrium price $\bar{p}$, making the statement of Theorem 3 not completely obvious. Moreover, $r(\delta, \eta)$ is increasing with δ confirming the intuition that under milder discounting we can afford waiting for better prices before harvesting. We observe that in the limiting case of $\delta = 1$ we have $r(1, \eta) = 1$, implying that when future is not discounted we should wait for prices at least as large as the equilibrium price to harvest.

In the general case when $\delta < 1$, $r(\delta, \eta) < 1$. Thus, if $p_t = \bar{p}$ then it is optimal to harvest everything available regardless of the values of η and δ. Another consequence is that for a small volatility, in the long run $p_t$ should be close to the equilibrium price $\bar{p}$, and condition (17) will most likely hold, implying that the GP is optimal. The larger the mean reversion rate, η, the sooner $p_t$ will fulfill condition (17). This shows some coherence with the deterministic case, studied by Rapaport et al. (2003), where the GP is optimal for all $t$ when prices are constant in time.

But Theorem 3 is not always useful. The most critical cases occur when $p_1$ is smaller than $r\bar{p}$ and the speed of mean reversion is slow (e.g., $\eta = 0.1$) or the volatility is high. Here, (17) is not likely to be satisfied and we do not know what is the optimal quantity to be harvested. In the next subsection, we present the results of some numerical experiments showing that Theorem 3 is a key ingredient to construct a policy that approximates the optimal policy extremely well.

### *3.2.2 Numerical experiments in the finite horizon case*

We use the censored recombining binomial tree proposed by Bastian-Pinto et al. (2009) to approximate the price process. Such model is proven to converge to a mean reverting process as time increments go to zero. Using Stochastic Dynamic Programming techniques, we obtain the optimal trajectory for every initial state.

We implement an alternative harvesting policy, using the threshold $r\bar{p}$ derived in (17) as a reservation price: we harvest everything available if $p_t$ is above $r\bar{p}$ and nothing if it is below. Theorem 3 assures that if (17) holds, then it is optimal to harvest everything

**Table 2** Comparison between the optimal, the $\Pi_0$ and the greedy policies

| (n, T) | (3, 10) | (4, 15) | (5, 15) | (6, 40) |
|---|---|---|---|---|
| Avg $\Pi_*$ | 7.78 | 7.80 | 6.49 | 8.09 |
| Avg $\Pi_0$ | 7.75 | 7.80 | 6.47 | 8.08 |
| Avg GP | 7.34 | 6.87 | 5.49 | 6.61 |
| Increment $\Pi_* : \Pi_0$ (%) | 0.33 | $<10^{-5}$ | 0.38 | 0.13 |
| Increment $\Pi_0 : GP$ (%) | 5.4 | 13.6 | 17.9 | 22.2 |
| Increment $\Pi_* : \Pi_0$ for $\mathbb{X}^S$ (%) | 0.11 | $<10^{-4}$ | 0.93 | 0.17 |
| Increment $\Pi_0$ : GP for $\mathbb{X}^S$ (%) | 12.9 | 55.3 | 11.7 | 54.0 |

available but we have no theoretical justification for postponing the harvest if it does not. We call this the alternative policy $\Pi_0$ and compare the present value it provides with that of the optimal policy $\Pi_*$ and with that of the GP. As in the numerical experiments presented in § 3.1.4, we also report the percentage increment change for the particular state $\mathbb{X}_1 = (0, 0, \ldots, S) := \mathbb{X}^S$.

Table 2 shows statistics of the present value of all three policies for $\delta = 0.95$, $\eta = 1.3$, $\sigma = 1.2$ and $S = 1$ with initial price $p_1 = 2$ and $\bar{p} = 1$. For every pair $n$ and $T$ we compute the average over all initial states of the optimal present value and the present value obtained with policies $\Pi_0$ and GP. The percentage increment (Avg optimal/Avg $\Pi_0 - 1$) is always less than 0.5 %, showing that $\Pi_0$ is nearly optimal for this set of parameters. The comparison for the $\mathbb{X}^S$ state yields similar results: the percentage increment is less than 1.0 % for all instances. The performance of the GP is quite different: the average percentage increment (Avg $\Pi_0$/Avg GP $- 1$) for adopting $\Pi_0$ instead of GP can be more than 50 % for two of the parameters' choices.

Our main numerical finding for the O–U case is that although we do not know exactly what the optimal policy is, the proposed $\Pi_0$ policy performs extremely well and it is straightforward to implement in practice. The experiment also highlights the poor performance of the GP, showing that the reservation price is indeed relevant and harvesting everything available instead of postponing the harvest for a later period can lead to extremely suboptimal policies.

## 4 Conclusions

We study a harvest scheduling problem under price uncertainty. At every period a decision maker must decide the amount of timber that is going to be harvested. In this work we considered discrete versions of the two most common stochastic processes used to model timber prices: the geometric Brownian motion and the Ornstein-Uhlenbeck process.

We adopted the model described in Rapaport et al. (2003), in which harvest can only occur at a maturity age $n$ and beyond maturity. In the deterministic case, when prices are constant, Rapaport et al. showed that the greedy policy is always optimal. The most important contribution of our work is the characterization of the optimal policy under stochastic prices. Put it simply, when prices are random it is unclear if one should harvest everything available at once or postpone the decision in the hope that prices will rise.

When prices follow a geometric Brownian motion, we obtained a closed expression (16) characterizing optimality of the greedy policy that depends only on the drift of the process and on the discount factor. Furthermore, when the condition is not satisfied we also characterized the optimal policy and proved it is not greedy, showing a clear depart from the deterministic case.

When prices follow an Ornstein-Uhlenbeck process, we were able to obtain a new theoretical condition which implies optimality of the greedy policy. It is interesting to note that such condition depends on timber price at every period $t$ and can be interpreted as a *reservation price*. When the condition is not satisfied, we do not know what is the optimal policy but our numerical experiments showed that a policy based on such reservation prices achieves benefits within 1 % of the optimal solution.

Although several simplifications were made mainly for tractability, we believe our results provide important managerial insights for the harvest scheduling problem. The majority of works that incorporate uncertain elements present numerical results that provide intuition about such problems but do not offer complete descriptions of the optimal policy. Our theoretical results give a complete understanding of the geometric Brownian motion case and characterize the optimal policy for the Ornstein-Uhlenbeck case by providing a closed expression for a reservation price.

There are very few benchmark problems in stochastic harvest scheduling and we hope our results will help future applied research in the field. For instance, the calibration of more complex models could benefit from our theoretical results in the sense that the optimal policy obtained numerically for simpler versions of some model should be greedy if the appropriate condition on the parameters is satisfied.

Future work includes a more complex forest growth model. In this article it is assumed that prices only are stochastic, treating tree growth as deterministic for all ages. In fact, forest growth dynamics is very simply modeled, considering only the age of the trees and neglecting natural mortality or the occurrence of rare catastrophic events that may affect the forest. To take into account natural mortality a very simple modification in the matrix that defines the growth dynamics is needed. In addition, it is assumed that there is no natural regeneration, and that every bit of liberated surface is planted immediately with seedlings of the same species.

Another interesting avenue of research is the incorporation of risk measures in the objective function. The maximization of expected value does not protect the decision maker against the possibility of extreme events such as a low-probability scenario with extremely low prices. There are several risk measures available in the literature such as the Value-at-Risk, Conditional Value-at-Risk, Mean Semi-deviation and others and theoretical advances have made possible the incorporation of such measures within the optimization context. It would be interesting to analyze the structure of the optimal policy if the objective function was for instance, a weighted combination of the expected value and some risk measure.

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## Appendix: Proofs of the results

*Proof of Lemma 1* Since prices are positive, we can use the Monotone Convergence Theorem to exchange the order of the expected value and the limit:

$$\begin{aligned}V(\mathbb{X}_t,p_t) &= \sup_{\Pi}\mathbb{E}_{|p_t}\left[\lim_{T\to\infty}\sum_{l=t}^{T}\delta^{l-t}p_l\pi_l(\mathbb{X}_l,p_l)\right]\\ &= \sup_{\Pi}\lim_{T\to\infty}\mathbb{E}_{|p_t}\left[\sum_{l=t}^{T}\delta^{l-t}p_l\pi_l(\mathbb{X}_l,p_l)\right].\end{aligned}$$

By linearity of integrals and using that the total area available is $S$, we have

$$\begin{aligned}\sup_{\Pi}\lim_{T\to\infty}\left[\sum_{l=t}^{T}\delta^{l-t}p_t e^{(l-t)\mu}\pi_l(\mathbb{X}_l,p_l)\right] &\le \lim_{T\to\infty}Sp_t\left[\sum_{l=t}^{T}\delta^{l-t}p_t e^{(l-t)\mu}\right]\\ &\le Sp_t\frac{1}{1-\delta e^{\mu}}<\infty.\end{aligned}$$

□

*Proof of Theorem 1* To prove that the greedy policy is optimal, we check that the benefit associated with it ($Q^{GP}$) satisfies the Bellman equation (15) from any initial condition. To find the expression of $Q^{GP}$ we start by finding the harvests associated to the greedy policy, $c_t$ for all $t\in\mathcal{T}$. If $\mathbb{X}_t=(\bar{x},x_n,\ldots,x_2,x_1)$ the first $n$ harvests will be:

$$\begin{cases}c_t=\bar{x}+x_n,\\ c_{t+1}=x_{n-1},\\ \quad\vdots\\ c_{t+n-1}=x_1.\end{cases}$$

Observe that at time $t+n$ the mature trees are exactly $\bar{x}+x_n$, hence $c_{t+n}=c_t$. From then on, the harvests start repeating with period $n$:

$$c_{t+in+j}=c_{t+j},\quad \text{for all } i \quad \text{and for all } j=0,\ldots,n-1 \quad \text{s.t. } (t+in+j)\in\mathcal{T}.$$

We consider first the infinite horizon case. With the $c_t$ found above we get,

$$\begin{aligned}Q_t^{GP}(\mathbb{X},p_t) &= \mathbb{E}_{|p_t}\left[\sum_{i=0}^{\infty}\left(\delta^{in}p_{t+in}\bar{x}+\sum_{j=0}^{n-1}\delta^{in+j}p_{t+in+j}x_{n-j}\right)\right]\\ &= p_t\sum_{i=0}^{\infty}\left((\delta e^{\mu})^{in}\bar{x}+\sum_{j=0}^{n-1}(\delta e^{\mu})^{in+j}x_{n-j}\right).\end{aligned}\tag{18}$$

If $c_t=c\in[0,CA\mathbb{X}_t]$, then the state at $t+1$ is $\mathbb{X}_{t+1}=(\bar{x}+x_n-c,x_{n-1},\ldots,x_1,c)$ and the corresponding expected benefit when GP is applied from $t+1$ onwards is

$$Q_t^{GP}(\mathbb{X}_{t+1},p_{t+1})=p_{t+1}\sum_{i=0}^{\infty}\left((\delta e^{\mu})^{in}(\bar{x}+x_n-c)+\sum_{j=0}^{n-2}(\delta e^{\mu})^{in+j}x_{n-j-1}+(\delta e^{\mu})^{in+n-1}c\right).$$

Inserting $V = Q^{GP}$ into the right hand side of the Bellman equation (15), the argument of the max operator is

$$\begin{aligned}\Phi(c) &= p_t c + \delta \mathbb{E}_{|p_t}\left[p_{t+1}\sum_{i=0}^{\infty}\left((\delta e^{\mu})^{in}(\bar{x}+x_n-c)+\sum_{j=0}^{n-2}(\delta e^{\mu})^{in+j}x_{n-j-1}+(\delta e^{\mu})^{in+n-1}c\right)\right]\\ &= p_t c + p_t\sum_{i=0}^{\infty}\left((\delta e^{\mu})^{in+1}(\bar{x}+x_n-c)+\sum_{j=0}^{n-2}(\delta e^{\mu})^{in+j+1}x_{n-j-1}+(\delta e^{\mu})^{in+n}c\right).\end{aligned} \tag{19}$$

The coefficient affecting $c$ in $\Phi(c)$ is

$$\begin{aligned}\operatorname{coeff}(c) &= p_t\left(1-\sum_{i=0}^{\infty}(\delta e^{\mu})^{in+1}+\sum_{i=0}^{\infty}(\delta e^{\mu})^{in+n}\right) = p_t\left(\sum_{i=0}^{\infty}(\delta e^{\mu})^{in}-\sum_{i=0}^{\infty}(\delta e^{\mu})^{in+1}\right)\\ &= (1-\delta e^{\mu})\sum_{i=0}^{\infty}(\delta e^{\mu})^{in} > 0.\end{aligned}$$

As $\operatorname{coeff}(c) > 0$, the maximum in (15) is attained when $c = CA\mathbb{X}_t = \bar{x}+x_n$. Inserting this value of $c$ in (19) yields

$$\begin{aligned}\Phi(\bar{x}+x_n) &= p_t\left((\bar{x}+x_n)+\sum_{i=0}^{\infty}\left(\sum_{j=0}^{n-2}(\delta e^{\mu})^{in+j+1}x_{n-j-1}+(\delta e^{\mu})^{in+n}(\bar{x}+x_n)\right)\right)\\ &= p_t\sum_{i=0}^{\infty}\left((\delta e^{\mu})^{in}(\bar{x}+x_n)+\sum_{j=1}^{n-1}(\delta e^{\mu})^{in+j}x_{n-j}\right) = Q^{GP}(\mathbb{X}_t, p_t),\end{aligned}$$

showing that (15) holds and that the GP is optimal.

The proof for the finite horizon case is very similar but more involved. Indeed, let $t$ be expressed as $T-(kn+l)$, where $k = \lfloor\frac{T-t}{n}\rfloor$ and $l$ takes one value in $\{0,\ldots,n-1\}$. This way of expressing $t$ puts in evidence that after completing $k$ cycles there will be still $l \in [0,\ldots,n-1]$ time steps to go until reaching the end of the horizon. The expected benefit associated with GP is

$$Q_t^{GP}(\mathbb{X}_t, p_t) = p_t\left[\sum_{i=0}^{k-1}\delta^{in}e^{in\mu}\left(\bar{x}+\sum_{j=0}^{n-1}\delta^j e^{j\mu}x_{n-j}\right)+\delta^{kn}e^{kn\mu}\left(\bar{x}+\sum_{j=0}^{l}\delta^j e^{j\mu}x_{n-j}\right)\right].$$

Again, we need to check that $Q^{GP}$ satisfies (15) and that the maximum is attained for $c = CA\mathbb{X}_t$. To this end, it is convenient to divide the study into two cases, depending on the value of $l$: (a) $l > 0$ and (b) $l = 0$ (see the proof of Theorem 2). In both cases, the verification of (15) follows exactly the same lines that the proof in the infinite horizon case. We leave the details to the reader. □

*Proof of Lemma 2* Given any initial state $\mathbb{X} = (\bar{x}, x_n, \ldots, x_1)$, it is obvious that al least one of its coordinates must be strictly positive. Without loss of generality, we assume that $x_n > 0$. We compare the value function with the benefit delivered by the GP (see (18)) to obtain,

$$V_t(\mathbb{X}, p_t) \geq Q_t^{GP}(\mathbb{X}, p_t) \geq p_t \sum_{i=0}^{\infty} (\delta e^{\mu})^{in} x_n = \infty.$$

□

*Proof of Theorem 2* To prove that the AP is optimal, we check that the expected benefit associated with it ($Q^{AP}$) satisfies the Bellman equation (15). After some computations we can prove that if $\mathbb{X}_t = (\bar{x}, x_n, \ldots, x_1)^T$ and $t = T - (kn + j)$, with $k = \lfloor \frac{T-t}{n} \rfloor$ and $l$ takes one value in $\{0, \ldots, n-1\}$, the total actualized benefit is

$$\begin{aligned} Q_t^{AP}(\mathbb{X}_t, p_t) &= \mathbb{E}_{|p_t}\left[\delta^l p_{t+l}\left(\bar{x} + \sum_{j=0}^{l} x_{n-j}\right) + \sum_{i=1}^{k} \delta^{in+l} p_{t+in+l} S\right] \\ &= p_t\left[(\delta e^{\mu})^l \left(\bar{x} + \sum_{j=0}^{l} x_{n-j}\right) + \sum_{i=1}^{k} (\delta e^{\mu})^{in+l} S\right]. \end{aligned} \tag{20}$$

For the rest of the proof we divide the study into two cases depending on the value of $l$: (a) $l > 0$ and (b) $l = 0$.

(a) Case $t = T - (kn + l)$, with $l > 0$. Here, $t + 1 = T - (kn + l - 1)$ with $l - 1 = 0, \ldots, n-2$. Denoting $c = c_t$, we get that $Q_{t+1}^{AP}(A\mathbb{X}_t + Bc, p_{t+1})$ can be expressed as

$$Q_{t+1}^{AP}(A\mathbb{X}_t + Bc, p_{t+1}) = p_{t+1}\left[(\delta e^{\mu})^{l-1}\left(\bar{x} + x_n - c + \sum_{j=0}^{l-1} x_{n-j-1}\right) + \sum_{i=1}^{k} (\delta e^{\mu})^{in+j-1} S\right].$$

Inserting $V = Q^{AP}$ into the right-hand side of the Bellman equation (15), the argument of the max operator is

$$\begin{aligned} \Phi(c) &= p_t c + \delta \mathbb{E}_{|p_t}\left[-p_{t+1}\left[(\delta e^{\mu})^{l-1}\left(\bar{x} + x_n - c + \sum_{j=0}^{l-1} x_{n-j-1}\right) + \sum_{i=1}^{k} (\delta e^{\mu})^{in+l-1} S\right]\right] \\ &= p_t c - p_t\left[(\delta e^{\mu})^l\left(\bar{x} + x_n - c + \sum_{j=0}^{l-1} x_{n-j-1}\right) + \sum_{i=1}^{k} (\delta e^{\mu})^{in+l} S\right] \\ &= p_t c[1 - (\delta e^{\mu})^j] + Q_t^{AP}(\mathbb{X}_t, p_t). \end{aligned}$$

As the coefficient of $c$ is negative, the maximum is attained when $c = 0$ and $\Phi(0)$ is exactly $Q_t^{AP}(\mathbb{X}_t, p_t)$, showing that the Bellman equation (15) holds.

(b) Case $t = T - kn$. In this case, we have $t + 1 = T - [(k-1)n + n - 1)]$ and denoting $c = c_t$ $Q_{t+1}^{AP}$ is expressed as

$$\begin{aligned} Q_{t+1}^{AP}(A\mathbb{X}_t + Bc, p_{t+1}) &= p_{t+1}\left[(\delta e^{\mu})^{n-1}\left(\bar{x} + x_n - c + \sum_{l=0}^{n-2} x_{n-l-1} + c\right) + \sum_{i=1}^{k-1} (\delta e^{\mu})^{in+n-1} S\right] \\ &= p_{t+1}\left[(\delta e^{\mu})^{n-1} S + \sum_{i=2}^{k} (\delta e^{\mu})^{in-1} S\right] = p_{t+1} \sum_{i=1}^{k} (\delta e^{\mu})^{in-1} S. \end{aligned}$$

Inserting $V = Q^{AP}$ into the right-hand side of the Bellman equation (15), the argument of the max operator is

$$\Phi(c) = p_t c + \delta \mathbb{E}_{|p_t}\left[p_{t+1}\sum_{i=1}^{k}(\delta e^{\mu})^{in-1}S\right].$$

The coefficient of $c$ is positive, and thus, the maximum is attained when $c = \bar{x} + x_n$. So we have,

$$\Phi(\bar{x}+x_n) = p_t(\bar{x}+x_n) + p_t\sum_{i=1}^{k}(\delta e^{\mu})^{in}S.$$

The right hand side is exactly (20) when $j = 0$, hence we have $\Phi(\bar{x}+x_n) = Q_t^{AP}(\mathbb{X}_t, p_t)$ and the Bellman equation is satisfied.

In both cases, we have shown that $Q_t^{AP}(\cdot,\cdot)$ satisfies the Bellman equation, hence it is the value function and the proposed policy is optimal. □

Before presenting the proof of Theorem 3 we present a technical proposition that will be used in the proof of the theorem.

**Proposition 1** *Let* $r_{mj} = 1 - \frac{1-\delta^j}{e^{-m\eta}(1-\delta^j e^{-j\eta})}$ for $m, j \in \mathbb{N}$ *with* $\delta \in (0,1)$ *and* $\eta > 0$. Then[4]

$$\frac{\delta(1-e^{-\eta})}{(1-\delta e^{-\eta})} = r_{01} \geq r_{mj}, \quad \text{for all } m \geq 0,\, j \geq 1$$

*with equality only if* $m = 0$ *and* $j = 1$.

*Proof* First observe that

$$r_{01} = 1 - \frac{1-\delta}{1-\delta e^{-\eta}} = \frac{\delta(1-e^{-\eta})}{1-\delta e^{-\eta}}.$$

Hence the following are equivalent,

$$\begin{array}{rccc} & r_{01} & \geq & r_{mj} \\ \iff: & \frac{1-\delta}{1-\delta e^{-\eta}} & \leq & \frac{1-\delta^j}{e^{-m\eta}(1-\delta^j e^{-j\eta})} \\ \iff: & e^{-m\eta}\frac{1-\delta^j e^{-j\eta}}{1-\delta e^{-\eta}} & \leq & \frac{1-\delta^j}{1-\delta} \\ \iff: & e^{-m\eta}\sum_{i=0}^{j-1}(\delta e^{-\eta})^i & \leq & \sum_{i=0}^{j-1}\delta^i. \end{array}$$

Given that $e^{-\eta} < 1$ the last inequality is always valid.

The equality can only hold if $m = 0$, which implies that $e^{-m\eta} = 1$, and $j = 1$, which gives that the two sums comprise only one term and are equal to one. □

*Proof of Theorem 3* We present a proof under the assumption that at every step we either harvest nothing at all or everything available. The result is still valid without this assumption, but we decided not to present the general proof for two reasons: (i) the general proof follows the same lines that the one presented here, but the calculus are more cumbersome; (ii) due to the linearity of the forestry model, this assumption is equivalent to requiring that the coefficient of $c$ in (15) is never zero, but having a zero coefficient is an

[4] A stronger result holds: $r_{m,j} \geq r_{m',j'}$ for all $m \leq m'$ and $j \leq j'$, with equality iff $m = m'$ and $j = j'$. For the proof of Theorem 3 the weaker version presented is enough.

event with zero probability.

The main idea of the proof is to consider the role played by $c$ in all the possible expressions of $V_t(\cdot,\cdot)$. Of course, characterizing completely every possible expression of $V_t(\cdot,\cdot)$ is a titanic task, but we will only be interested in the coefficient affecting $c$.

After harvesting $c_t = c$ the state $\mathbb{X}_{t+1}$ is

$$\mathbb{X}_t = \begin{pmatrix} \bar{x}_t \\ x_{n,t} \\ x_{n-1,t} \\ \vdots \\ x_{1,t} \end{pmatrix} \longrightarrow \mathbb{X}_{t+1} = A\mathbb{X}_t + Bc = \begin{pmatrix} \bar{x}_t + x_{n,t} - c \\ x_{n-1,t} \\ x_{n-2,t} \\ \vdots \\ c \end{pmatrix},$$

and the resulting Bellman equation (15) can be stated in terms of $\mathbb{X}_t$ and $c$ as follows,

$$V_t(\mathbb{X}_t, p_t) = \max_c \{p_t c + \delta \mathbb{E}_{|p_t}[V_{t+1}(A\mathbb{X}_t + Bc, p_{t+1})]\}.$$

To simplify the notation we denote $\Phi(c)$ the argument of the maximum, this is,

$$\Phi(c) = p_t c + \delta \mathbb{E}_{|p_t}[V_{t+1}(A\mathbb{X}_t + Bc, p_{t+1})].$$

From $t+1$ on, two different situations must be considered: (i) nothing is harvested in the next $n$ steps or (ii) the first harvest occurs at $t + j_0$ with $j_0 \in \{1, \ldots, n-1\}$.

In case (i), the state at $t+n$ will be

$$\mathbb{X}_{t+n} = (S - c, c, 0, \ldots\ldots, 0)^T.$$

It is easy to see that the influence of $c$ extinguishes as the constraint on $c_{t+n}$ is $c_{t+n} \le S$. We do not know the complete expression of $V_{t+1}(\cdot,\cdot)$ but we do know that the coefficient of $c$ in $\Phi(c)$ will be simply $p_t$.

If on the contrary, we are in case (ii), there will be harvest at $t + j_0$ with $j_0 \in \{1, \ldots, n-1\}$ and a term $\delta^{j_0} \mathbb{E}_{|p_t}[-p_{t+j_0}]$ will be added to the coefficient of $c$ in $\Phi(c)$,

$$\Phi(c) = p_t c + \delta^{j_0} \mathbb{E}_{|p_t}[p_{t+j_0}(\bar{x}_t + x_{n,t} + \ldots + x_{n-j_0,t} - c) + \delta V_{t+j_0+1}(\cdot,\cdot)].$$

We are left with the task of characterizing the coefficient of $c$ in $V_{t+j_0+1}(\cdot,\cdot)$. We know that $c$ only affects two coordinates of the state $\mathbb{X}_{t+j_0+1}$, and these two coordinates are: $x_{t+j_0+1,j_0+1} = c$ and $x_{t+j_0+1,1} = \bar{x}_t + x_{n,t} + \ldots + x_{n-j_0,t} - c$. From now on, we will omit the state variables and will represent the state as

$$\mathbb{X}_{t+j_0+1} = (0, *, \ldots\ldots, *, c, *, \ldots\ldots, *, * - c)^T. \tag{21}$$

Until $x_{t+j_0+1,j_0+1} = c$ becomes available for harvesting, the coefficient of $c$ in $\Phi(c)$ will not be affected by any of the actions taken. These trees will reach maturity exactly at $t+n$, when the state will be

$$\mathbb{X}_{t+n} = (*, c, *, \ldots\ldots, *, * - c, \ldots\ldots, *)^T,$$

with $x_{t+n,n-j_0} = * - c$. Once we reach this time step, two different situations must be considered: (i) nothing is harvested until we reach the time step $t + n + j_0$ or (ii) the first harvest occurs at $t + m_1$ with $m_1 \in \{n, \ldots, n + j_0 - 1\}$.

In case (i), we would have $\bar{x}_{t+n+j_0} = * + c$ and $x_{t+n+j_0,n} = * - c$ and the influence of $c$ vanishes. We know that the coefficient of $c$ in $\Phi(c)$ will be simply $p_t + \delta^{j_0}\mathbb{E}_{|p_t}[-p_{t+j_0}]$.

In case (ii), there is harvest at $t + m_1$ and a new term is added to the coefficient of $c$ in $\Phi(c)$: $\delta^{m_1}\mathbb{E}_{|p_t}[p_{t+m_1}]$. After the harvest, the state will be

$$\mathbb{X}_{t+m_1+1} = (0, *, \ldots\ldots, * - c, *, \ldots\ldots, *, * + c)^T.$$

Again, there will be some time steps where coefficient of $c$ in $\Phi(c)$ will not be affected by any of the actions taken. In fact, until the fraction of trees $* - c$ reaches maturity no new terms of the coefficient are generated.This will happen at $t + n + j_0$, when the state would be in this case

$$\mathbb{X}_{t+j_0+n} = (*, * - c, *, \ldots\ldots, *, * + c, \ldots\ldots, *)^T.$$

As before, two different situations must be considered: (i) nothing is harvested until we reach the time step $t + m + n$ or (ii) the first harvest occurs at $t + m_1 + j_1$ with $j_1 < n$.

In case (i), we will have $\bar{x}_{t+m+n} = * - c$ and $x_{t+m+n,n} = * + c$ and the influence of $c$ extinguishes. The coefficient of $c$ in $\Phi(c)$ will be simply:
$p_t + \delta^{j_0}\mathbb{E}_{|p_t}[-p_{t+j_0}] + \delta^{m_1}\mathbb{E}_{|p_t}[p_{t+m_1}]$.

In case (ii), there is harvest at $t + m_1 + j_1$ and the term $\delta^{m_1+j_1}\mathbb{E}_{|p_t}[-p_{t+m_1+j_1}]$ is added to the coefficient of $c$.

Observe that the state at the following step $(t + m_1 + j_1 + 1)$ is

$$\mathbb{X}_{t+m_1+j_1+1} = (0, *, \ldots\ldots, * + c, *, \ldots\ldots, *, * - c)^T.$$

The state at $t + m_1 + j_1 + 1$ is of the same form that the state at $t + j_0 + 1$ (see (21)) and the same reasoning can be applied over and over. In each completed cycle first the term "$* + c$" is harvested, the index $m_i$ is generated, with $m_i \geq m_{i-1} + n$, and the term $\delta^{m_i}\mathbb{E}_{|p_t}[p_{t+m_i}]$ is added to the coefficient of $c$ in $\Phi(c)$. Secondly,the term "$* - c$" is harvested, the index $j_i$ is generated, with $1 \leq j_i < n$ and the term $\delta^{m_i+j_i}\mathbb{E}_{|p_t}[-p_{t+m_i+j_i}]$ is added to the coefficient.

The algorithm ends either at the end of the interval or before that if we reach a state of the form

$$\mathbb{X}(t + m_i + n) = \begin{pmatrix} * - c \\ * + c \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} \text{ or } \mathbb{X}(t + m_i + j_i + n) = \begin{pmatrix} * + c \\ * - c \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}$$

and the influence of $c$ extinguishes as $CA\mathbb{X}(\cdot)$ is independent of $c$.

The coefficient of $c$ is the finite sum of terms of the form

$$\delta^{m_i}\mathbb{E}_{|p_t}[p_{t+m_i}] - \delta^{m_i+j_i}\mathbb{E}_{|p_t}[p_{t+m_i+j_i}]. \tag{22}$$

plus possibly one positive term $\delta^{m_i}\mathbb{E}_{|p_t}[p_{t+m_i}]$. The proof is completed by showing that the coefficient of $c$ is positive. To this end, we study the sign of expression (22) and its relation with condition (17),

$$\begin{aligned}
&0 \le \delta^{m_i}\mathbb{E}_{|p_t}[p_{t+m_i}] + \delta^{m_i+j_i}\mathbb{E}_{|p_t}[-p_{t+m_i+j_i}]\\
\Longleftrightarrow:\ &0 \le \mathbb{E}_{|p_t}[p_{t+m_i}] + \delta^{j_i}\mathbb{E}_{|p_t}[-p_{t+m_i+j_i}]\\
&= e^{-m_i\eta}p_t + (1-e^{-m_i\eta})\bar{p} - \delta^{j_i}[e^{-(m_i+j_i)\eta}p_t + (1-e^{-(m_i+j_i)\eta})\bar{p}]\\
&= e^{-m_i\eta}p_t(1-\delta^{j_i}e^{-j_i\eta}) + \bar{p}[1-\delta^{j_i} - e^{-m_i\eta}(1-\delta^{j_i}e^{-j_i\eta})]\\
\Longleftrightarrow:\ &\frac{p_t}{\bar{p}} \ge 1 - \frac{1-\delta^{j_i}}{e^{-m_i\eta}(1-\delta^{j_i}e^{-j_i\eta})}.
\end{aligned}$$

But the right hand side above is exactly $r_{m_i j_i}$ of Proposition 1. This proposition together with Condition (17) imply

$$\frac{p_t}{\bar{p}} \ge r_{01} \ge r_{m_i j},$$

which proves that (22) is non-negative. Furthermore, we know that the second inequality holds strictly unless $m_i = 0$ and $j = 1$. Hence, the positivity of the coefficient of $c$ in (15) follows for every case, except only for the case where it consists exclusively of one term of the form (22) with $m_i = 0$ and $j = 1$, i.e.,

$$p_t - \mathbb{E}_{|p_t}[\delta p_{t+1}]. \tag{23}$$

From the previous calculus it is direct to see that (23) is positive if (17) holds strictly. If, on the contrary, (17) holds with equality, there is no influence of $c$ in the value of $V_t(\mathbb{X}_t, p_t)$ and we can freely chose the value of $c$ provided that it is feasible. We impose $c_t = CA\mathbb{X}_t$, which concludes the proof. □

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