The stochastic Mitra-Wan forestry model: risk neutral and risk averse cases

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Abstract

We extend the classic Mitra and Wan forestry model by assuming that prices follow a geometric Brownian motion. We move one step further in the model with stochastic prices and include risk aversion in the objective function. We prove that, as in the deterministic case, the optimal program is periodic both in the risk neutral and risk averse frameworks, when the benefit function is linear. We find the optimal rotation ages in both stochastic cases and show that they may differ significantly from the deterministic rotation age. In addition, we show how the drift of the price process affects the optimal rotation age and how the degree of risk aversion shortens it. We illustrate our findings for an example of a biomass function and for different values of the model's parameters.

Keywords: Forestry· dynamic programming· risk analysis· coherent risk measures

JEL Classification: Q23 · C61 · D81

1 Introduction

In his seminal paper of 1849, Martin Faustmann (1849) states the problem of finding the economic value of an even-aged forest stand whose economic value depends only on its age. Considering an infinite horizon discrete time model and periodic policies, he obtains an expression for the present value of the stand, a formula that can be used to determine an optimal harvesting age. This optimal harvesting age is now known as the Faustmann rotation age. The elegance and simplicity of this result stems from the fact that we deal with a forest of identically aged trees. The generalization of the optimal rotation problem to a forest with many even-aged stands was already considered at that time, but its complete resolution remains open even today.

In two remarkable papers in the mid-eighties, Mitra and Wan (1985, 1986) reconsider the optimal harvesting problem as a dynamic optimization problem, considering multiple stands of even aged trees that can be partially or totally

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harvested at every time period. If the utility function is linear, they prove that harvesting every tree at the Faustmann rotation age or beyond is optimal.

All the above mentioned papers deal with the deterministic case. Our main contribution in this work is to drop the deterministic assumption of the Mitra and Wan model and to consider stochastic prices. We study the case in which timber prices follow a geometric Brownian motion (GBM). The GBM is widely used in finance to model stock prices, and in particular in the famous Black and Scholes model.

In forestry, it is common to use GBM to model timber prices. In Alvarez and Koskela (2007a) and Clarke and Reed (1989), the problem of determining the optimal harvesting age of an even-aged forest is considered when stock size is stochastic and prices follow a GBM. In Clarke and Reed (1989), the authors solve the single-rotation problem theoretically as an optimal stopping problem, and propose a numerical iterative scheme to solve the ongoing-rotation problem. Alvarez and Koskela (2007a) only deal with the single-rotation problem and study how the optimal policy varies with price and stock size volatility. The work of Thomson (1992) also extends Faustmann's classical model assuming prices follow a GBM. The author solves the problem numerically in a discrete time framework. Recently, in Di Corato et al. (2013), forest land conversion to other uses is studied when the value of the forest's environmental services provided follow a GBM. Other works as Penttinen (2006); Yoshimoto and Shoji (1998) also support the use of GBM to model price dynamics and characterize the corresponding optimal harvesting policy numerically.

The migration from deterministic to stochastic models raises the question of how to deal with the uncertainties in the model. The classical approach is to maximize *expected* benefits, leading to a risk-neutral model. More recently, there has been research beyond the risk neutral framework in order to account for risk aversion. In most cases results show differences in the optimal policies when the objective function is no longer the expected value. Interest in understanding the effects of risk aversion in finance and economics has increased significantly in the last years, specially in dynamic models (see, for example, Blomvall and Shapiro (2006); Denuit et al. (2006); Mitra and Roy (2012); Riedel (2004)). In this paper, risk aversion is represented by the Conditional Value-at-Risk (CVaR), a risk measure widely used in finance that was popularized by Rockafellar and Uryasev (2000). The CVaR protects the decision maker against extreme losses and satisfies the axioms of coherence defined in Artzner et al. (1999).

The literature of risk averse models in forestry is still scarce. In Gong and Löfgren (2003), the authors investigate the effect of risk aversion on a two-period harvest investment decision model and show that the optimal harvesting policy depends on the sign of a marginal variance function. Alvarez and Koskela (2006) shows that higher risk aversion shortens the expected rotation period and that increased forest value volatility decreases the optimal harvesting threshold, which is not true under risk neutrality. More importantly, in Alvarez and Koskela (2007b) the authors are the first to consider risk aversion in stochastic ongoing rotation models. In that work, a study of the impact of taxation on the optimal rotation period is presented both in the risk neutral and risk averse frameworks. It is shown that under risk aversion the optimal harvesting threshold is lower, which translates into a shorter rotation period. The effect of risk aversion on the length of the rotation period is also studied in Gong and Löfgren (2008), using a model with regeneration costs. In this case, the risk averse rotation period can be either longer or shorter than the risk neutral one, depending on these costs. It is worth mentioning that all the papers cited in this paragraph consider single stand models, while we work with a multi-stand forestry model.

We believe this is one of the first papers in the literature to incorporate coherent risk measures into forestry. The majority of papers in forestry that model risk aversion use the expected utility criterion. Although risk measures and expected utility criteria are related through dual representations of the risk functional (see example 6.14 of Shapiro et al. (2009)), both have its own advantages and drawbacks. The main difficulties associated with the expected utility criterion is the choice of which expected utility better captures the preferences of decision makers and the calibration of the parameters (Delbaen et al., 2011; Peters, 2012; Schechter, 2007). Gong and Löfgren (2008) propose to use the mean-variance rule of Markowitz (1952) or stochastic dominance analysis when the utility function is not available. In this context, risk measures can be regarded as an alternative approach that addresses some of the difficulties associated with the expected utility. A comparison of both approaches is out of the scope of this paper.

The paper has two goals. *First*, we extend the classic Mitra and Wan linear forestry model by assuming that prices are not deterministic. In particular, we show that if prices follow a GBM then the optimal harvesting policy consists in harvesting trees whose ages are greater than or equal to the rotation period. As in Mitra and Wan (1985, 1986), the rotation period can be found by maximizing a known function over all age classes. *Second*, we move one step further in the model with stochastic prices and include risk aversion in the objective function. Using the CVaR as our risk measure, we fully characterize the optimal harvesting policy and provide comparisons between the deterministic, risk neutral and risk averse formulations. In particular, we show how an increase in the degree of risk aversion shortens the optimal rotation length. In passing we characterize the variation of the optimal rotation length with the discount factor also in the deterministic case, a question that has not been tackled in the literature.

The paper is organized as follows: Section 2 presents the forest growth model and the price process we are considering. Section 3 contains the main contributions of this work: we state our optimization problem and provide an explicit characterization of the optimal harvesting policies in the stochastic risk neutral and risk averse frameworks. In Section 4 we compare the different harvesting policies and discuss how they are affected by changes in the drift and variance of the price process as well as the discount factor and the level of risk aversion. Section 5 presents a numerical example to illustrate the theoretical results obtained. Finally, Section 6 concludes the paper and brings up some futures lines of research. All the proofs are relegated to the Appendix.

2 The model

We consider a discrete time model for the optimal management of a forest that was proposed by Mitra and Wan (1985, 1986) and further investigated by Salo and Tahvonen (2002, 2003). The model is built under the following assumptions:

- The timber content of a tree depends only on the age of that tree, i.e., growth is a pure aging process.
- There are no costs of harvesting, plantation or maintenance.
- n is the age immediately after which trees die. [1]
- The number of the trees in a given piece of land is proportional to its area.

For each time period $t \in I\!N$, we denote $x_{a,t} \geq 0$, $a = 1, \ldots, n$, the surface area occupied by trees of age a by the end of time period t. We represent the state of the forest by the vector [2] $\boldsymbol{x}_t = (x_{1,t}, ..., x_{n,t})$.

At the end of every time period two things happen: first the planner must decide how much land to harvest of every age-class, $\boldsymbol{c}_t = (c_{1,t}, ..., c_{n,t})$, where $c_{a,t} \in [0, x_{a,t}]$, and all the cleared land is replanted with seedlings of the same species. [3] Hence, the forest always covers completely the available land that we assume to be homogeneous and of normalized unit size, thus, the state of the forest belongs to the $(n-1)$-dimensional simplex $\Delta = \{\boldsymbol{x} \in I\!R^n_+ : \sum_a x_a = 1\}$.

The remaining trees of age class a after the harvest will comprise the $(a+1)-$th age class by the end of the following time period, i.e.,

$$x_{a+1,t+1} = x_{a,t} - c_{a,t}. \tag{1}$$

Given that n is the age after which trees die, we assume that $c_{n,t} = x_{n,t}$ for all t, without loss of optimality. Summarizing, at time period $t+1$ the state is

$$\boldsymbol{x}_{t+1} = \Big(\sum_{a=1}^{n} c_{a,t},\ x_{1,t} - c_{1,t}, ..., x_{n-1,t} - c_{n-1,t} \Big).$$

According to the description above, we provide the conditions that a *program* (i.e., a feasible trajectory) must fulfill.

[1] However, one difference should be noted. In their treatment, Mitra-Wan take n to be the age at which the timber volume coefficient per unit of land is maximized, claiming that "for any reasonable objective function for the economy, trees will never be allowed to grow beyond age n" (Mitra and Wan, 1986, p. 332). It was pointed out by Khan and Piazza (2012), that the concavity of the benefit function favors a homogeneously configured forest and that it may be optimal to postpone harvesting beyond age n in order to reshape the forest into a more homogeneous state. Following Khan and Piazza (2012), we circumvent this by assuming n to be the age after which a tree dies.

[2] The expressions in bold print represent vectors.

[3] Under the hypotheses that there are no costs of harvesting or plantation, it is optimal to plant all the available land at every time period.

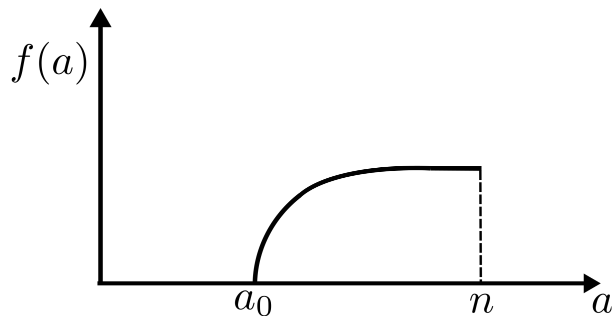

Figure 1: Biomass function

Definition 2.1 *A sequence $\{\boldsymbol{x}_t\}_{t\in I\!N}$ is called a* program *if, for each t,*

$$\begin{cases} \boldsymbol{x}_t \in \Delta, \\ x_{a+1,t+1} \leq x_{a,t} \quad a = 1,\ldots,n-1. \end{cases}$$

Associated to any program there is a unique sequence of non-negative harvesting decision vectors, $\{\boldsymbol{c}_t\}_{t\in I\!N}$, that can be calculated directly using (1): $c_{a,t} = x_{a+1,t+1} - x_{a,t}$ for $a = 1,\ldots,n-1$ and $c_{n,t} = x_{n,t}$ for all t.

We represent the timber content of a unit of area covered by trees of age a by the timber volume coefficients f_a for each age $a = 1,\ldots,n$. For a harvest of $\boldsymbol{c}_t = (c_{1,t}, ..., c_{n,t})$, the timber volume obtained to sell in the market is given by $c_t = \sum_a f_a c_{a,t}$.

We assume that there is $a_0 \geq 0$ such that $f_a = 0$ for all $a \leq a_0$, and that the timber volume coefficients satisfy [4]

$$f_{a-1} + f_{a+1} < 2f_a \qquad \forall a = 2,\ldots,n-1, \quad a \geq a_0. \qquad \text{(Assumption 1)}$$

For some of our results, we need a stronger assumption: the timber volume coefficients are given by $f_a = f(a)$ where f is called *the biomass function*, and it is such that $f : [0,n] \to I\!R_+$ and

$$\begin{array}{ll} (i) & f(a) = 0 \text{ for } a \leq a_0 \text{ for some } a_0 \geq 0, \text{ and} \\ (ii) & f(a) \text{ is twice differentiable and concave} \\ & \text{for } a_0 \leq a \leq n. \end{array} \qquad \text{(Assumption 2)}$$

An example of such function is given in Figure 1. Comparing Assumption 2 with the assumptions on the biomass function made on Mitra and Wan (1985), we see that we are adding differentiability and eliminating monotonicity.

Example 2.1 *We present a toy example illustrating the concept of a program. Assume that $n = 4$ and that we always harvest every tree that is three years old*

[4]In particular, this property is satisfied by any discrete mid-point concave function, see Murota (2003).

or older and nothing else. If the initial state is

$$\boldsymbol{x}_0 = (0.1,\ 0.3,\ 0.2,\ 0.4),$$

the following state will be

$$\boldsymbol{x}_1 = (0.6,\ 0.1,\ 0.3,\ 0),$$

and from then on a periodic sequence follows:

$$\begin{aligned}
\boldsymbol{x}_2 &= (0.3,\ 0.6,\ 0.1,\ 0)\\
\boldsymbol{x}_3 &= (0.1,\ 0.3,\ 0.6,\ 0)\\
\boldsymbol{x}_4 &= (0.6,\ 0.1,\ 0.3,\ 0)\\
\boldsymbol{x}_5 &= (0.3,\ 0.6,\ 0.1,\ 0)\\
\boldsymbol{x}_6 &= (0.1,\ 0.3,\ 0.6,\ 0)\\
&\vdots
\end{aligned}$$

In the first time period, the harvest is $\boldsymbol{c}_0 = (0,\ 0,\ 0.2,\ 0.4)$ *and the total timber volume obtained is* $c_0 = f_3 \times 0.2 + f_4 \times 0.4$*. In the following time periods the total timber volumes obtained are:*

$$c_1 = f_3 \times 0.3, \quad c_2 = f_3 \times 0.1, \quad c_3 = f_3 \times 0.6, \quad c_4 = f_3 \times 0.3, \quad c_5 = f_3 \times 0.1.$$

Throughout this article, we will prove that for our optimization problem the optimal harvesting policy belongs to a particular type of harvesting policies: the ones where every tree above a certain age θ is harvested at *every* time period (like the one on the example above). These policies yield $\theta-$periodic programs. In these cases, the harvesting age θ is also called the *rotation age.*

2.1 Price process

Several alternatives have been offered in the literature to model timber prices. Due to its analytic tractability, its popularity in options theory and its consistency with an informationally efficient market, the geometric Brownian motion (GBM) has been extensively used in the forest literature, as discussed in the Introduction. Other authors claim that several historical timber prices exhibit a reversion to the mean, and the Ornstein-Uhlenbeck (O-U) is the preferred process in those cases (Insley (2002), Gjolberg and Guttormsen (2002)).

In Dixit and Pindyck (1994) the authors discuss GBM and O-U as two possible candidates for modelling commodity prices. However, statistical tests cannot distinguish between the two, and a significant amount of historical data might be necessary to reach a conclusion. In the works Plantiga (1998),Tahvonen and Kallio (2006), Piazza and Pagnoncelli (2014), the authors consider both GBM and O-U and obtain numerical and theoretical results results for the two processes.

Recent works have considered other processes and more complex models. For example in Chen and Insley (2012) timber prices are represented by a regime switching model that allows the parameters of the process to change over time, according to a Markov state variable. In Khajuria, Kant and Laaksonen-Craig (2009) the authors use statistical tests applied to real-world data to justify the need for considering jump processes.

Econometric and equilibrium models are also very popular tools to model timber prices. In Zhou and Buongiorno (2006) the authors propose an econometric model in which space-time correlations between prices in neighboring regions are used as price predictors. In Mei, Clutter and Harris (2010) the authors show, by applying various time-series techniques, that the vector autoregressive model provides the best forecasting for a 1-year period under the mean absolute percentage error criterion. The recent paper of Latta, Sjølie, Solberg (2013) reviews the most important partial equilibrium models used nowadays in Forestry, in which commodity prices are determined endogenously.

The focus of this paper is to characterize optimal programs in the risk neutral and risk averse cases, finding closed form expressions for the rotation ages. Motivated by its extensive use in the forestry literature and by its analytical tractability, we chose the GBM to represent the price dynamics. Since the process is well-known, we define it without further details:

$$dp_t = \mu p_t dt + \sigma p_t dW_t, \tag{2}$$

where $\mu \in \mathbb{R}$ is the *drift* of the GBM, $\sigma > 0$ is the constant variance, and W_t denotes the Wiener process. Timber price evolves continuously, that is, at every time t the price is observable, but harvest decisions can only be made at pre-specified times $t = 1, 2$, etc. In the next section we will derive the optimal programs for the model assuming prices follow a GBM, both for the risk neutral and risk averse cases.

3 Optimal programs

In the risk neutral setting, sequential stochastic decision problems are often stated in a nested formulation setting (see, for example, Birge and Louveaux (1997)), which is essentially a dynamic programming formulation of the problem. Building on this framework, Ruszczyński and Shapiro (2006) extended the classical conditional expectation to the more general *conditional risk mappings*, which accommodate several risk measures in addition to the expected value. Denoting by $\rho_{|p_t}[\cdot]$ the conditional risk mapping for $t \in \mathbb{N}$, the stochastic dynamic model we will consider in this work can be formulated as follows:

$$\begin{cases} \min_{c_0} & \Big\{ - p_0 c_0 + \delta \rho_{|p_0} \Big[\min_{c_1} \Big\{ - p_1 c_1 + \delta \rho_{|p_1} \Big[\min_{c_2} \Big\{ \dots \Big\} \Big] \Big\} \Big] \Big\} \\ \text{s.t.} & c_t = \sum_{a=1}^{n-1} f_a c_{a,t} + f_n x_{n,t} \quad \forall t, \\ & c_{a,t} = x_{a,t} - x_{a+1,t+1} \qquad a = 1, \dots, n-1 \quad \forall t, \\ & \{\boldsymbol{x}_t\} \text{ is a program}, \\ & \boldsymbol{x}_0 \text{ given}, \end{cases} \tag{3}$$

where $\delta \in (0,1)$ is the discount factor. [5] When $\rho_{|p_t}[\cdot]$ is equal to the conditional expectation $\mathbb{E}_{|p(t)}[\cdot]$ we fall into the risk neutral case. With other conditional risk mappings we fall into the risk averse case, in which the decision maker wishes to be protected against extreme losses, represented here by low prices.

There are several possible choices for the conditional risk mapping ρ and the choice is usually guided by the type of risk behavior to be captured and by mathematical tractability. An important class of risk measures that has gained popularity after the publication of Artzner et al. (1999) are the so-called *coherent risk measures*. A measure is said to be coherent if it satisfies four axioms: positive homogeneity, translation invariance, monotonicity and subadditivity.

A tractable coherent risk measure that is widely used in the financial context is the Conditional Value-at-Risk (CVaR). For continuous random variables (for general distributions we refer the reader to Rockafellar and Uryasev (2000, 2002)), the CVaR_α of level $\alpha \in [0,1)$ can be defined in terms of the more popular VaR_α as follows:

$$\begin{aligned} \text{VaR}_\alpha[X] &= \min\{t \in \mathbb{R} : P(X \le t) \le 1 - \alpha\}, \\ \text{CVaR}_\alpha[X] &= \mathbb{E}[X | X > \text{VaR}_\alpha[X]]. \end{aligned}$$

In summary, the CVaR_α is the average of losses greater than the α-quantile. Aside from coherence, the CVaR has several desirable properties that justify its popularity: it is continuous with respect to the risk averse level α, it is consistent with the classical mean-variance approach, it is easy to optimize even for non-normal distributions and, for a convex combination of random variables, it is convex with respect to the weights. Finally, in a static context, the minimization of the CVaR can be well approximated by a linear program, which is easy to solve by commercial solvers (Rockafellar and Uryasev, 2000).

It is well-known that $\mathbb{E}_{|p_t}[p_{t+1}] = e^{\mu} p_t$ when prices evolve according to a GBM process. Furthermore, p_{t+1} conditional on p_t follows a lognormal distribution, and it is possible to compute

$$\text{CVaR}_\alpha[p_{t+1} | p_t] = e^{\mu} \, \frac{\Phi(\Phi^{-1}(\alpha) - \sigma)}{\alpha} \, p_t,$$

where σ is the constant variance of the price process defined in (2) and Φ is the cumulative distribution function of the normal random variable with mean

[5] In Lemma 3.1 we prove that the nested objective function is well-defined in the cases we are interested in.

0 and variance 1. [6] We denote

$$\kappa = \frac{1}{\alpha}\Phi(\Phi^{-1}(\alpha) - \sigma), \tag{4}$$

and observe that $\kappa \leq 1$.

For deterministic and constant prices, this problem was solved in Mitra and Wan (1985). Our optimization problem generalizes the classical Mitra and Wan (MW) formulation. Indeed, both formulations have the same constrains and only differ on the objective function. In addition, observe that in the particular case of deterministic and constant prices (p), our objective function in (3) reduces to $\sum_{t=0}^{\infty} \delta^t p c_t$, [7] which is exactly the MW objective function.

When prices follow a GBM we know that $\mathbb{E}_{|p_t}[p_{t+1}] = e^{\mu} p_t$, and we can write the objective function of (3) as

$$\sum_{t \in \mathbb{N}} \delta^t e^{\mu t} c_t p_0 = \sum_{t \in \mathbb{N}} r^t c_t p_0, \quad \text{with} \quad r = \delta e^{\mu}.$$

Furthermore, using that the conditional CVaR is positive homogeneous and that $\text{CVaR}_{p_t}[p_{t+1}] = e^{\mu} \kappa p_t$ when prices follow a GBM, we can write the objective function as

$$\sum_{t \in \mathbb{N}} \delta^t e^{\mu t} \kappa^t c_t p_0 = \sum_{t \in \mathbb{N}} r^t c_t p_0 \quad \text{with} \quad r = \delta e^{\mu} \kappa.$$

In Mitra and Wan (1985), the authors show that there is an age θ such that the optimal policy consists in harvesting every tree of age θ or older, letting all the younger trees grow. Such strategy is called Faustmann policy. It is also shown that the optimal harvesting age θ is such that

$$\frac{f_\theta \delta^\theta}{1 - \delta^\theta} \geq \frac{f_a \delta^a}{1 - \delta^a} \qquad \forall a = 1, \dots, n. \tag{5}$$

This policy produces a program $\{\boldsymbol{x}_t\}_{t \in \mathbb{N}}$ that is θ-periodic from $t = 2$ onward. In the following theorem we show that the optimal harvesting policies for the risk neutral and risk averse stochastic harvesting problem have the same structure, with a different rotation period. From now on, we denote the optimal harvesting age when prices are deterministic and constant by θ_d, to distinguish it from the optimal harvesting ages in the stochastic cases. Before stating the theorem, we present a technical lemma assuring that the nested objective function (3) is finite in the cases we deal with.

Lemma 3.1 *The objective function in* (3) *when* $\rho = \mathbb{E}$ *(resp.* $\rho = CVaR_\alpha$*) is finite whenever* $\delta e^{\mu} < 1$ *(resp.* $\delta e^{\mu} \kappa < 1$*).*

Theorem 3.1 *Under Assumption 1, the optimal harvesting policy for problem* (3) *when* $\rho = \mathbb{E}$ *and* $\delta e^{\mu} < 1$ *(resp.* $\rho = CVaR_\alpha$ *and* $\delta e^{\mu} \kappa < 1$*) consists in harvesting every tree of age* $\theta_{\mathbb{E}}$ *(resp.* θ_ρ*) or older at every time period, leaving*

[6] The computations can be found in Pagnoncelli and Piazza (2012).
[7] We have $\rho(a) = a$ for any constant a.

all the younger trees uncut. Furthermore, the optimal harvesting ages $\theta_{\mathbb{E}}$ and θ_ρ are determined by

$$\frac{f_{\theta_{\mathbb{E}}}(\delta e^{\mu})^{\theta_{\mathbb{E}}}}{1-(\delta e^{\mu})^{\theta_{\mathbb{E}}}} \geq \frac{f_a(\delta e^{\mu})^a}{1-(\delta e^{\mu})^a} \qquad \forall a = 1,\dots,n, \tag{6}$$

$$\frac{f_{\theta_\rho}(\delta e^{\mu}\kappa)^{\theta_\rho}}{1-(\delta e^{\mu}\kappa)^{\theta_\rho}} \geq \frac{f_a(\delta e^{\mu}\kappa)^a}{1-(\delta e^{\mu}\kappa)^a} \qquad \forall a = 1,\dots,n. \tag{7}$$

The proof is presented in the Appendix.

The similarities between inequalities (5), (6) and (7) are evident: in order to find the optimal harvesting age we solve three instances of the problem

$$\theta(r) = \arg\max\left\{\frac{f_a r^a}{1-r^a}, a = 1,\dots,n\right\}, \tag{8}$$

with different values of r:

- $r = \delta$ if prices are deterministic and constant,
- $r = \delta e^{\mu}$ if prices follow a GBM process and the decision maker is risk neutral and
- $r = \delta e^{\mu}\kappa$ if prices follow a GBM process and the decision maker is risk averse.

Given the importance of (8) for the deterministic and stochastic cases, we study how its solution varies with r when $r \in (0,1)$. If the solution of (8) is not unique, we choose the highest one. [8] To this end, we focus first in the continuous version of our discrete optimization problem, assuming that age is a continuous variable,

$$\text{Max}\left\{F_r(a) = \frac{f(a)r^a}{1-r^a}, 1 \leq a \leq n\right\}, \tag{9}$$

where $r \in (0,1)$ and f satisfies Assumption 2.

The next theorem, whose proof is in the appendix, characterizes the solution of (9) and will be useful to compare the optimal harvesting ages of the deterministic and stochastic cases.

Theorem 3.2 *Under Assumption 2, the maximum of* (9) *is attained at a unique point, $a^* := a^*(r)$, and $a^*(r)$ is increasing with r. Furthermore, $a^*(r)$ is the unique point in the interval (a_0, n) where $F'_r(a) = 0$.*

[8] In Mitra and Wan (1985), the authors assume that the solution of (5) is unique. For some values of r, (8) may have multiple solutions. We will see later that Assumption 2 implies that (8) has at most two solutions that happen to be consecutive. In such a case, both Faustmann policies yield exactly the same value for Problem (3) for every initial condition. We take the convention of choosing the largest value as the solution.

The fact that a^* is the unique local extreme of $F_r(a)$ also implies that $F_r(a)$ is non-decreasing on $[a_0, a^*]$ and non-increasing on $[a^*, n]$. In consequence, the optimum of the discrete version (8) is either $\lfloor a^* \rfloor$ or $\lfloor a^* \rfloor + 1$. [9] As we said above, if the values of (8) for $\lfloor a^* \rfloor$ and $\lfloor a^* \rfloor + 1$ coincide, we choose $\lfloor a^* \rfloor + 1$ according to our convention. We can show a property of $a^*(r)$ weaker than the monotonicity of $\theta(r)$ using Theorem 3.2:

$$r' \geq r \Rightarrow a^*(r') \geq a^*(r) \Rightarrow \theta(r') \geq \theta(r) - 1.$$

If, besides, $\lfloor a^*(r') \rfloor \geq a^*(r)$ then $\theta(r') \geq \theta(r)$.

4 Discussion and insights

Deterministic vs. risk neutral cases: if the drift of the process is positive ($\mu > 0$), i.e., if expected prices tend to grow, we have $e^{\mu} > 1$, which by Theorem 3.2 implies that $\theta_d < \theta_{\mathbb{E}}$. Hence, the rotation age is larger in the stochastic risk neutral case than in the deterministic case. The intuition is that if prices are random but are expected to grow, then the risk neutral decision maker should always harvest older trees when comparing his policy with the deterministic case. If $\mu < 0$, prices are expected to decrease over time and therefore the rotation age is shorter in the stochastic case than in the deterministic case.

A financial interpretation can be obtained by representing the discount factor δ by the continuously-compounded discount rate $e^{-\eta}$, where η represents the interest rate. In other words, one dollar today is worth e^{η} dollars in the next period. We have

$$r = \delta e^{\mu} = e^{-\eta} e^{\mu} = e^{-\eta+\mu} = \tilde{\delta}, \quad (10)$$

which tells us that the stochastic risk neutral problem can be solved like a deterministic one with constant prices but with a different discount rate $r = \tilde{\delta}$, which is affected by the drift of the price process.

Risk neutral vs. risk averse cases: the constant r in (8) for the risk averse case has an extra multiplicative factor κ, defined in (4). Such factor characterizes risk averse behavior: as κ is always less than one, we have that the optimal rotation age for the risk averse case, θ_{ρ}, is always shorter than the risk neutral optimal rotation age. Furthermore, (6) and (7) provide a way to estimate by how much the rotation age is shortened due to risk aversion. This is rather intuitive: a risk averse manager will always tend to harvest trees before a risk neutral one in order to be protected against low prices, even though she might be giving up a higher future yield.

For values of σ close to zero, that is, in the case where timber prices are less volatile, the constant κ is close to one and the two solutions become essentially equivalent. For larger values of σ, that is, for higher volatility, we observe that κ is closer to zero and θ_{ρ} could be significantly shorter than $\theta_{\mathbb{E}}$. We will present numerical illustrations of such behavior in the next section.

[9] Where $\lfloor a \rfloor$ stands for the integer part of a.

Let us now turn our attention to the role of the risk aversion parameter α. As we increase α, that is, as decision makers become less risk averse, the multiplicative factor κ increases. In the limiting case, where $\alpha = 1$, it can be seen that $\kappa = 1$ and the risk neutral policy is retrieved. On the other hand, values of α close to zero yield significant differences in the optimal harvesting age between the two cases, capturing the risk attitude of the decision maker.

Undiscounted benefits, deterministic and stochastic cases: the deterministic case when future benefits are not discounted (i.e., $\delta = 1$) is studied in Mitra and Wan (1986). Although the value function in (3) is infinite when $\delta = 1$ and prices are constant, the problem can be tackled using the concepts of *maximal* and *optimal* programs. Such concepts are based in studying the long run behavior of the accumulated differences of benefits of every pair of programs.[10] It is proved in Mitra and Wan (1986) that the optimal harvesting policy also consists in harvesting every tree of a certain age or beyond, and that this optimal harvesting age, denoted by θ_1, is such that

$$\frac{f_{\theta_1}}{\theta_1} \geq \frac{f_a}{a} \qquad \forall a = 1, \ldots, n. \tag{11}$$

In the stochastic risk neutral problem, when the interest rate η equals the drift of the price process μ (see (10)), we have $\delta e^{\mu} = e^{-\eta} e^{\mu} = 1$ and the $\theta_{\mathbb{E}}$ is given by (11), implying that the optimal harvesting policy of the discounted stochastic risk neutral problem coincides with the one of the undiscounted deterministic problem in this case. Analogously, in the stochastic risk averse problem θ_ρ is given by (11) whenever $\delta e^{\mu} \kappa = 1$.

Sensitivity of the rotation age with respect to δ: the three values of r considered ($\delta, \delta e^{\mu}$ and $\delta e^{\mu} \kappa$) are increasing with δ. Hence, by Theorem 3.2, the values of $\theta_d, \theta_{\mathbb{E}}$ and θ_ρ tend to increase with an increase of the discount factor. The variation of θ_d with respect to δ has not been previously studied in the literature.

5 Numerical illustrations

We illustrate our findings for a biomass function described in Salo and Tahvonen (2003):

$$f(a) = \frac{500}{1 + 40e^{-.048a}}. \tag{12}$$

Function (12) is first convex and then concave. Following what is done in Salo and Tahvonen (2003), we generate a timber volume coefficients vector by evaluating the function every 10 years during a 200-year period. Additionally we assume that the convex part generates zero benefit. The first 6 points lie in the

[10] For formal definitions and a brief discussion of multiple existing criteria we refer the reader to Khan and Piazza (2012).

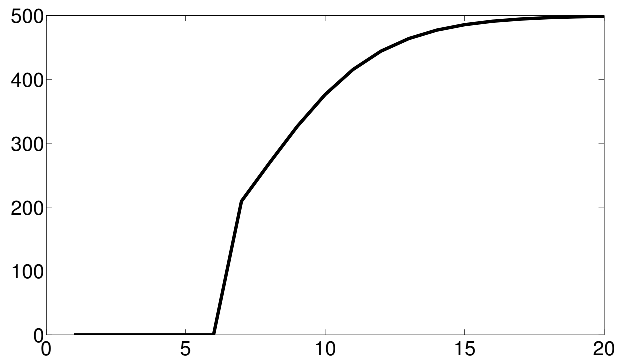

Figure 2: Discrete biomass function

convex part of the function so we assign value zero to the first 6 timber volume coefficients, i.e., $a_0 = 6$. The remaining 14 points are in the concave part of the function so we simply compute their values according to function (12). The resulting discretized vector f_d, illustrated in Figure 2, satisfies Assumption 1 for all $a \geq a_0$:

$$f_d = [0, 0, 0, 0, 0, 0, 209, 268, 326, 376, 415, 444, \\ 463, 476, 485, 490, 494, 496, 497, 498].$$

Table 1 shows the effect of the drift μ in the optimal rotation ages of the deterministic and risk neutral cases. As predicted by the theory, in the first row we have that the optimal rotation age of the deterministic case is smaller than the one for the risk neutral case (9 versus 11) because the drift of the price process μ is positive. When we decrease the drift μ from .1 to $.03, \theta_{\mathbb{E}}$ decreases because prices are expected to have only a moderate growth in the future. Moreover, in the second row we observe that the value of μ is positive but so small that the optimal harvesting ages $\theta_{\mathbb{E}}$ and θ_d coincide. In the last row we observe that a negative drift leads to $\theta_{\mathbb{E}} < \theta_d$, as opposed to the first row.

μ	θ_d	$\theta_{\mathbb{E}}$
.1	9	11
.03	9	9
$-.05$	9	8

Table 1: θ_d and $\theta_{\mathbb{E}}$ with $\delta = .9, \sigma = .1$ and $\alpha = .15$.

In Table 2 we compare the effect of changes in the risk aversion parameter α for the risk neutral and risk averse cases. When $\alpha = .8$, $\theta_{\mathbb{E}}$ and θ_ρ are similar (11 and 10) because the degree of risk aversion is close to one. As we decrease α

to .5, θ_ρ becomes equal to 9 and in the extreme risk averse case of $\alpha = .01$ the optimal age decreases to 7, which is the lowest age that has a timber volume coefficient greater than zero.

α	$\theta_{\mathbb{E}}$	θ_ρ
.8	11	10
.5	11	9
.01	11	7

Table 2: $\theta_{\mathbb{E}}$ and θ_ρ with $\delta = .9, \mu = .1$ and $\sigma = .1$.

Changes in the variance have a similar effect. We observe in Table 3 that an increase in the variance leads to a decrease in the optimal harvesting age because there is more variability on timber price. Starting with a value of $\sigma = .01$, we observe that the risk neutral and risk averse optimal harvesting ages coincide, that is, even for a very high level of risk aversion we do not observe differences between the two policies. Multiplying the variance by 5 decreases θ_ρ from 10 to 9. For a value ten times higher than the base value .01 the optimal harvesting age drops to 7, meaning the decision maker is willing to compromise gains in order to be protected against the volatility of timber prices.

σ	$\theta_{\mathbb{E}}$	θ_ρ
.01	10	10
.05	10	9
.10	10	7

Table 3: $\theta_{\mathbb{E}}$ and θ_ρ with $\delta = .85, \mu = .13$ and $\alpha = .1$.

6 Conclusions

In this paper we generalize the classic Mitra and Wan forestry model by considering stochastic timber prices. We model price dynamics through a geometric Brownian motion and study both the risk neutral and risk averse formulations. We show that, analogously to the deterministic case, the optimal program is periodic in both frameworks, but the rotation ages might be different. The optimal rotation age for the stochastic cases can be explicitly found by a simple rule, analogous to the one of the deterministic case. We study the sensitivity of the optimal rotation ages with respect to the parameters of the model. Finally, we indicate when the solution to the undiscounted deterministic problem is retrieved as the solution to a discounted stochastic problem.

Opposed to Thomson (1992), who obtains that higher volatility leads to higher benefits, our optimal program and total expected benefit in the risk neutral framework do not depend on the volatility of the price process. This can be explained by the differences of the models. When prices are too low, the decision maker in Thomson (1992) can dedicate land to an alternative use, protecting herself against extreme losses and obtaining better performance overall. Introducing the possibility of an alternative use of land in our model will open the possibility of a fairer comparison to other existing results.

Future work also includes considering a more complex forest growth model. For example, with a very simple modification in the growth dynamics it is possible to take into account natural mortality. However, proofs may need a significant adaptation.

Considering multi-species forests would allow us to compare our results with an important part of the present literature. The traditional paradigm indicates that a risk averse decision maker would favor a more homogeneous land allocation, establishing more ecologically friendly forests. Indeed, through numerical experiments, Hildebrandt et al. (2010), Knoke et al. (2005) and Roessiger et al. (2011) show that higher risk aversion implies higher mixture of species.

In this work we captured risk aversion through a risk measure: another possibility would be via expected utility theory, as proposed by Von Neumann and Morgenstern (2007). The connection between the two approaches has been investigated in Follmer and Schied (2004), and to study the differences between the two methodologies in the context of forestry using the Mitra and Wan forest as the benchmark model is an interesting line of future research.

7 Acknowledgements

This research was partially funded by Basal project CMM, Universidad de Chile. A. Piazza acknowledges the financial support of FONDECYT under project 1140720 and of CONICYT Anillo ACT1106. B. K. Pagnoncelli acknowledges the financial support of FONDECYT under projects 1120244 and 11130056.

Appendix

Proof of Lemma 3.1. Given any program $\{\boldsymbol{x}_t\}$ (not necessarily optimal) and $\{c_t\}$ the corresponding sequence of harvesting, we consider the finite horizon value of problem (3):

$$Q_T(\{\boldsymbol{x}_t\}) = -p_0c_0 + \delta\rho_{|p_0}[-p_1c_1 + \cdots + \delta\rho_{|p_{T-2}}[-p_{T-1}c_{T-1} + \delta\rho_{|p_{T-1}}[-p_Tc_T]]]$$

Due to the fact that $p_tc_t \geq 0$ for all t (when prices follow a GBM) and the monotonicity of any coherent risk measure we know that $Q_T(\{\boldsymbol{x}_t\}) \geq Q_{T+1}(\{\boldsymbol{x}_t\})$, hence the sequence $Q_T(\{\boldsymbol{x}_t\})$ either converges to the limit or diverges to $-\infty$ when $T \to \infty$. We prove now that $Q_T(\cdot)$ is bounded below for all T and therefore it has a limit. If $\rho = \mathbb{E}$, we have $\rho_{p_{t-1}}[-p_tc_t] = -e^{\mu}p_{t-1}c_t$ and hence,

$$\begin{aligned} Q_T(\{\boldsymbol{x}_t\}) &= -p_0c_0 + \delta\rho_{|p_0}[-p_1c_1 + \cdots + \delta\rho_{|p_{T-2}}[-p_{T-1}(c_{T-1} - \delta e^{\mu}c_T)]] \\ &\vdots \\ &= -p_0\sum_{t=0}^{T}(\delta e^{\mu})^tc_t \geq -p_0\sum_{t=0}^{T}(\delta e^{\mu})^t = -p_0\frac{1-(\delta e^{\mu})^{T+1}}{1-\delta e^{\mu}} \end{aligned}$$

where the inequality follows because $c_t \leq 1$ for all t. If $\delta e^{\mu} < 1$ we get $Q_T > -p_0 \frac{1}{1-\delta e^{\mu}} > -\infty$ for all T. This implies that the sequence Q_T converges when T goes to infinity. This limit is the value associated with program $\{\boldsymbol{x}_t\}$ in the infinite time horizon formulation. As the bound does not depend on the particular program we get that the minimal value is also well defined and finite.

When $\rho = \mathrm{CVaR}_{\alpha}$ the proof is the same, substituting δe^{μ} by $\delta e^{\mu}\kappa$. ■

Proof of Theorem 3.1. We denote by $\{\boldsymbol{x}_t^*\}$ and $\{c_t^*\}$ (resp. $\{\boldsymbol{x}_t\}$ and $\{c_t\}$) the program and the harvesting sequence generated by the proposed optimal policy (resp. any alternative optimal policy) from the initial state $\boldsymbol{x}^o$.

We treat first the risk neutral case. We let r represent δe^{μ} and define the auxiliary constants

$$\tau_a = \frac{r^a}{1-r^a} \quad \text{with } a = 1\ldots,n \tag{13}$$

and $\boldsymbol{q} \in I\!R^{n+1}$ where

$$\boldsymbol{q} = \Big(0, \frac{f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}}}{\tau_1}, \frac{f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}}}{\tau_2}, \ldots\ldots, \underbrace{\frac{f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}}}{\tau_{\theta_{\mathbb{E}}-1}}}_{\theta_{\mathbb{E}}\text{-}th\ coord.}, f_{\theta_{\mathbb{E}}}, \ldots\ldots, f_n\Big).$$

We recall that n is the age after which die, hence, it is suboptimal to let trees grow beyond n and without loss of generality, we assume that $c_t^n = x_t^n$ for all $\{x_t\}$. To simplify the calculation below we add the auxiliary $(n+1)-$ coordinate to the state vector $\boldsymbol{x}_t$, assuming that $x_{n+1,t} = 0$ for all t.

Let us first show that

$$c_t + \sum_{a=1}^{n} q_a x_{a,t+1} - \frac{1}{r}\sum_{a=1}^{n} q_a x_{a,t} \leq \sum_{a=1}^{n} q_{a+1}\left(x_{a,t} - \frac{x_{a+1,t}}{r}\right) \qquad \forall t, \tag{14}$$

with equality iff $\{\boldsymbol{x}_t\} = \{\boldsymbol{x}_t^*\}$ and $\{c_t\} = \{c_t^*\}$. We have

$$\begin{aligned} c_t + \sum_{a=1}^{n} q_a x_{a,t+1} - \frac{1}{r}\sum_{a=1}^{n} q_a x_{a,t} \ &= \ c_t + \sum_{a=1}^{n} q_a\left(x_{a,t+1} - \frac{1}{r}x_{a,t}\right) \\ = \ \sum_{a=1}^{n} f_a(x_{a,t} - x_{a+1,t+1}) &+ \sum_{a=1}^{\theta_{\mathbb{E}}-1} \frac{f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}}}{\tau_a}\left(x_{a+1,t+1} - \frac{x_{a+1,t}}{r}\right) \\ &+ \sum_{a=\theta_{\mathbb{E}}}^{n-1} f_a\left(x_{a+1,t+1} - \frac{x_{a+1,t}}{r}\right) \\ = \ \sum_{a=1}^{\theta_{\mathbb{E}}-1}\Big[f_a(x_{a,t} - x_{a+1,t+1}) &+ \frac{f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}}}{\tau_a}\left(x_{a+1,t+1} - x_{a,t} + x_{a,t} - \frac{x_{a+1,t}}{r}\right)\Big] \\ &+ \sum_{a=\theta_{\mathbb{E}}}^{n} f_a\left(x_{a,t} - \frac{x_{a+1,t}}{r}\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{a=1}^{\theta_{\mathbb{E}}-1}\left[\frac{1}{\tau_a}(f_a\tau_a - f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}})(x_{a,t}-x_{a+1,t+1}) + \frac{f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}}}{\tau_a}\left(x_{a,t}-\frac{x_{a+1,t}}{r}\right)\right] \\
&\quad + \sum_{a=\theta_{\mathbb{E}}}^{n} f_a\left(x_{a,t}-\frac{x_{a+1,t}}{r}\right) \\
&= \sum_{a=1}^{\theta_{\mathbb{E}}-1}\frac{1}{\tau_a}(f_a\tau_a - f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}})(x_{a,t}-x_{a+1,t+1}) + \sum_{a=1}^{n} q_{a+1}\left(x_{a,t}-\frac{x_{a+1,t}}{r}\right)
\end{aligned}
$$

The first sum is always less than or equal to zero, with equality if and only if $x_{a,t} = x_{a+1,t+1}$ for $a < \theta_{\mathbb{E}}$ which is equivalent to $c_{a,t} = 0$ for all $a < \theta_{\mathbb{E}}$. Hence, we have obtained (14).

We focus now in getting a bound independent of $\boldsymbol{x}_t$. We claim that

$$
\sum_{a=1}^{n} q_{a+1}\left(x_a - \frac{x_{a+1}}{r}\right) \leq \frac{f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}}}{\tau_1}, \tag{15}
$$

for every state $\boldsymbol{x}$, with equality if and only if $x_a = 0$ for all $a > \theta_{\mathbb{E}}$. Indeed,

$$
\begin{aligned}
\sum_{a=1}^{n} q_{a+1}\left(x_a - \frac{x_{a+1}}{r}\right) &= \sum_{a=1}^{n} q_{a+1}x_a - \sum_{a=2}^{n+1} q_a\frac{x_a}{r} \\
&= q_2x_1 + \sum_{a=2}^{n}\left(q_{a+1}-\frac{q_a}{r}\right)x_a \\
&= \frac{f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}}}{\tau_1}x_1 + \sum_{a=2}^{\theta_{\mathbb{E}}} f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}}\left(\frac{1}{\tau_a}-\frac{1}{r\tau_{a-1}}\right)x_a + \sum_{a=\theta_{\mathbb{E}}+1}^{n}\left(f_a - \frac{f_{a-1}}{r}\right)x_a \\
&= \sum_{a=1}^{\theta_{\mathbb{E}}}\frac{f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}}}{\tau_1}x_a + \sum_{a=\theta_{\mathbb{E}}+1}^{n}\left(f_a-\frac{f_{a-1}}{r}\right)x_a,
\end{aligned}
$$

where in the last step we use the following equality:

$$
\frac{1}{\tau_a} - \frac{1}{r\tau_{a-1}} = \frac{1}{\tau_1}, \tag{16}
$$

which follows from (13). To finish the proof of (15) we only need to show that

$$
f_a - \frac{f_{a-1}}{r} \leq \frac{f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}}}{\tau_1} \text{ for all } a > \theta_{\mathbb{E}}. \tag{17}
$$

This is evidently true if the left hand side is negative, hence we will only deal with the cases where $f_a - \frac{f_{a-1}}{r} > 0$. This means, in particular, that we only need to consider values of a such that $f_a - f_{a-1} > 0$. Let a_M denote $\arg\max f_a$. Thanks to Assumption 1 we know that

$$
f_n - f_{n-1} < \cdots < f_{a_M+1} - f_{a_M} < 0 < f_{a_M} - f_{a_M-1} < \cdots < f_2 - f_1
$$

$$\Rightarrow \quad 0 < f_a - f_{a-1} < f_{a-1} - f_{a-2} < \frac{f_{a-1} - f_{a-2}}{r} \qquad \forall a \leq a_M$$

$$\Rightarrow \quad f_a - \frac{f_{a-1}}{r} < f_{a-1} - \frac{f_{a-2}}{r} \qquad \forall a \leq a_M.$$

Using (16) and the fact that $f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}} > f_{\theta_{\mathbb{E}}+1}\tau_{\theta_{\mathbb{E}}+1}$ we obtain

$$f_{\theta_{\mathbb{E}}+1} - \frac{f_{\theta_{\mathbb{E}}}}{r} < \frac{f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}}}{\tau_1},$$

which finishes the proof of (17) as well as (15).

Putting (14) and (15) together we readily get

$$c_t + \sum_{a=1}^{n} q_a x_{a,t+1} - \frac{1}{r}\sum_{a=1}^{n} q_a x_{a,t} \leq \frac{f_{\theta_{\mathbb{E}}}\tau_{\theta_{\mathbb{E}}}}{\tau_1} \qquad \forall t \tag{18}$$

In summary, we have that (14) and (18) are fulfilled along any feasible program. Besides, along the proposed optimal program (14) is satisfied with equality for all t and (18) is satisfied with equality for all $t \geq 2$.

To finish the proof we compare the benefits obtained by *any* program and the proposed optimal program up until some T. As prices follow a GBM we know that $\mathbb{E}_{|p_t}[p_{t+1}] = e^{\mu}p_t$ and we can write the objective function of (3) as $\sum_{t\in\mathbb{N}} \delta^t e^{\mu t} c_t p_0 = \sum_{t\in\mathbb{N}} r^t c_t p_0$ with $r = \delta e^{\mu}$.

Using (14) to bound the difference corresponding to $t = 1$ and (15) for the other terms we get

$$\sum_{t=1}^{T} r^t p_t c_t - \sum_{t=1}^{T} r^t p_t c_t^* \leq r^{T-1}\sum_{a=1}^{n} q_a (x_{a,T}^* - x_{a,T}) p_0$$

Given that the product $q'(\boldsymbol{x}_T^* - \boldsymbol{x}_T)p_0$ is bounded for all T and that obviously $r^{T-1} \to 0$ when $T \to \infty$ we get that

$$\sum_{t\in\mathbb{N}} r^t p_t c_t \leq \sum_{t\in\mathbb{N}} r^t p_t c_t^*$$

and hence the proposed harvesting policy is optimal.

The proof in the risk averse case follows analogously. Indeed, using that the conditional CVaR is positive homogeneous and that $\mathrm{CVaR}_{p_t}[p_{t+1}] = e^{\mu}\kappa p_t$ when prices follow a GBM, taking $r = \delta e^{\mu}\kappa$, we can write the objective function as $\sum_{t\in\mathbb{N}} \delta^t e^{\mu t}\kappa^t c_t p_0 = \sum_{t\in\mathbb{N}} r^t c_t p_0$. Hence, it suffices to take $r = \delta e^{\mu}\kappa$ and substitute $\theta_{\mathbb{E}}$ by θ_ρ throughout the proof.

■

Proof of Theorem 3.2. The function $F_r(a)$ is non-negative and twice differentiable in $[a_0, n]$. Its derivative is $F_r'(a) = \frac{r^a}{1-r^a}\left(f'(a) + \frac{f(a)\ln(r)}{1-r^a}\right)$ and, hence,

$$F_r'(a) = 0 \iff \frac{f'(a)}{f(a)} = \frac{\ln(1/r)}{1-r^a}. \tag{19}$$

A priori, there might be several points satisfying (19). We claim that it is unique. Indeed, evaluating F_r'' in a^* such that $F_r'(a^*) = 0$ we get

$$F_r''(a^*) = \frac{r^{a^*}}{1 - r^{a^*}} f''(a^*) - \frac{r^{a^*} \ln^2(r)}{(1 - r^{a^*})^2} \ f(a^*) < 0.$$

It implies that every zero of the first derivative is a local maximum and, as there cannot be two local maxima without a minimum, we conclude that the first derivative has a unique zero. We denote this point as a_r^* to point out its dependence with the parameter r.

To see the variation of a_r^* with respect to r let us consider (19). We know that $F_r'(a)$ is zero only at a_r^* and that $F_r''(a_r^*) < 0$, which implies that $F_r'(a) > 0$ whenever $a < a^*(r)$. Hence, we know that the graph of $\frac{f'(a)}{f(a)}$ is above the one of $\frac{\ln(1/r)}{1-r^a}$ whenever $a < a^*(r)$. While the left hand side of (19) is independent of r, its right hand side is decreasing with respect to r. From Figure 3 it is easy to see that a_r^* is increasing with r.

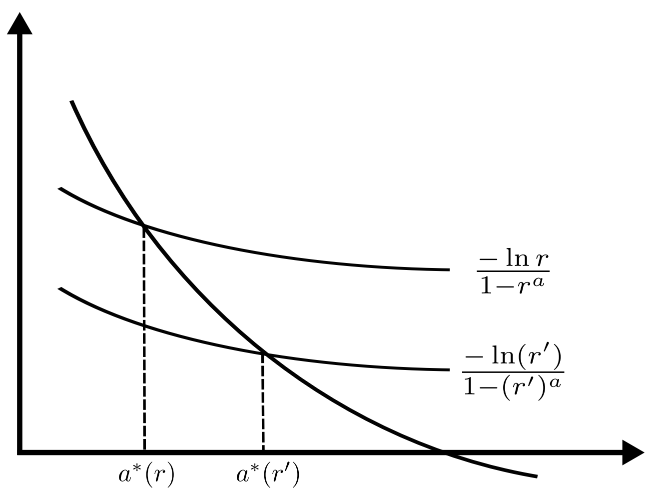

Figure 3: Monotonicity of $a^*(r)$ (with $r < r'$)

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