

Optimal control of renewable resources with alternative use*

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Abstract

We study the optimal harvesting of a renewable resource that cannot be continuously exploited, i.e. after the harvest the released space cannot be immediately reallocated to the resource. In the meantime, this space can be given an alternative use of positive utility. We consider a discrete time model over an infinite horizon with discounting, and characterize the optimality of two myopic policies, the greedy and the sustainable ones. We then show that the optimal strategy consists in applying one of these policies after a finite number of time steps. Value function and optimal feedback are explicitly determined.

Key words. Resource allocation, optimal control, infinite horizon, dynamic programming.

AMS Classification. 93C55, 93D20, 90C39, 91B76, 91B62, 90B50.

Introduction

We consider a discrete time modeling of the optimal harvesting of a renewable resource (that could typically be cultivated land, aqua-culture, forestry, ...) on a maximal space occupancy $S > 0$. We assume that the resource can be only harvested after reaching a “mature age” stage. In the general case this mature age could be arbitrary but in this work we will consider for sake of simplicity that the mature age is equal to one time step.

In this study, it is also assumed that the harvested space has to wait for at least one period before becoming available to be reallocated again to its primary use. This particular feature can occur from practical constraints such as land refreshing, cleaning, or other resting duties that do not allow an immediate re-use of the space for the resource. From the management point of view, unallocated space usually means economic loss. So we consider here that the unallocated space has a positive utility, that could be brought by an alternative use of the space compatible with the resting duties (this could be for instance tourism, scientific investigations or experimentations, ...). We also assume that this alternative use can be extended to more than one time step without any a priori time limit. The management problem is then to determine at each time step the best harvesting and arbitration between the resource allocation and the alternative use.

In this model we neglect the mortality of the resource in its mature age, arguing that a rational exploitation of the resource proceeds to spatial rotations so that no space stays occupied by the same unharvested resource for a long time.

Mathematical modeling of natural resources has received great interest since the pioneer work of C. Clark [2]. Today, theoretical questions in bio-economics of natural resources are commonly investigated with the help of dynamical systems models and optimal control theory [4, 5, 11, 12]. Special interest has been given to the sustainable management of natural renewable resources as part of a intense research around the concept of *sustainable development* [6, 7, 9]. The World Commission on Environment and Development (1987) produced the most quoted definition of sustainable development: “Economic and social development

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that meets the needs of the current generation without undermining the ability of future generations to meet their own needs”. There seems to be consensus about this definition but a lot of divergence on how to put the idea of sustainable development into operation. Among the different definitions of sustainability, we point out the one by Daily and Erlich [7]: “Sustainability characterizes any process or condition that can be maintained indefinitely without interruption, weakening, or loss of valued qualities”. Influenced by it, we pay special attention to management policies that keep a constant resource.

To the best of our knowledge, the question of alternative use of a resource have been scarcely studied within this framework. Some authors have tackled the problem of the optimal harvesting of a resource when there exists an alternative use of space, but without any constraint to space allocation. In [15], Salo and Tahvonen consider a multi-aged, one-species forest where an alternative use of the land is possible. They characterize the state whose optimal trajectory is constant, the sustainable state, prove that it is a local saddle point and give some numerical examples that exhibit convergence to such state. In [3], Cominetti and Piazza study a multi-species forest and prove that two types of asymptotic behavior are possible, either there is convergence to the sustainable state for every initial condition, or the optimal trajectory converges towards the set of states yielding a periodic optimal trajectory. This means that the choice of the transient stage (i.e. the anticipation decisions) is crucial before reaching a myopic strategy. Optimal periodic cycles appear as well in [14], where a simpler model of multi-aged, single species forest is considered. Assuming that harvest is restricted *a priori* to mature trees older than a certain age and that the growth after maturity is neglected, the authors show that every optimal trajectory becomes periodic in finite time. This implies that the problem can be reduced to a finite dimensional one and solved numerically.

Inspired by these ideas, we study in this paper two particular myopic strategies — the *constant policy* and the *greedy periodic policy* — analyzing the conditions under which they are optimal. These particular policies are the key ingredients for the characterization of the value function and optimal trajectories in the general case.

Our approach consists in determining first the subsets of initial conditions for which those myopic strategies are optimal. In a second stage, we prove that an optimal trajectory has to reach one of these subsets in finite time. Our proofs are entirely based on dynamic programming and concavity argumentations, which is not so often met in the literature (most of the time Maximum Principle or reformulation in terms of mathematical programming problems are used), apart from the analysis of pure growth models [17]. For instance, in [16] and [18], dynamic pricing (or “support prices”) are used to show the optimality of a periodic solution over infinite horizon for problems with similar structure. Dynamic programming aims at characterizing the value function, which is the unique solution of the Bellman equation. This is the combination of the Bellman equation with concavity properties that plays here a determining role for obtaining short and elegant proofs. As a comparison, we provide in an appendix alternative proofs based on mathematical programming and duality, adapted from more general results [3] to the present framework.

The paper is organized as follows. In §1 we present the optimization model to be solved, while in §2 we state and prove the main results of the paper. We first present necessary and sufficient conditions that make the myopic strategies (constant and greedy policies) optimal, and then give explicitly the value function and an optimal trajectory for any possible initial condition, solving completely the problem. Conclusions are dressed in §3. Finally, alternative proofs with the mathematical programming approach are given in Appendix.

1 The optimization problem

With no loss of generality we assume that $S = 1$. For each period $t \in \mathbb{N}$ we denote $z_t \geq 0$ the fraction of space occupied by the resource. We must decide how much resource u_t to harvest

$$0 \leq u_t \leq z_t$$

and the fraction v_t of the space occupied by the alternative use that will be reallocated to the resource

$$0 \leq v_t \leq 1 - z_t. \tag{1}$$

The constraints on u and v can then be expressed as

$$(u_t, v_t) \in C(z_t) \quad \text{where} \quad C(z) = [0, z] \times [0, 1 - z] \quad (2)$$

and the dynamics of the system are simply written as

$$z_{t+1} = z_t - u_t + v_t. \quad (3)$$

Given an initial condition z_0 , the objective to be maximized is the discounted total benefit

$$J(z_0, u, v) = \sum_{t=0}^{\infty} b^t [U(u_t) + W(1 - z_t)]$$

over all admissible controls (u, v) , where $U(\cdot)$ and $W(\cdot)$ are the utilities of the harvest and the use of the unallocated space respectively.

We assume the usual hypotheses on utility functions and discount factor. Let $\mathcal{C}_{[0,1]}$ be the set of C^1 non-decreasing concave functions from $[0, 1]$ to \mathbb{R}_+ .

Assumption A0. The functions $U(\cdot)$ and $W(\cdot)$ belong to $\mathcal{C}_{[0,1]}$. The discount rate b belongs to the interval $(0, 1)$.

We denote $\mathbf{z}, \mathbf{u}, \mathbf{v}$ the infinite sequences of states and controls. Admissible sequences have to fulfill the dynamics (3) and the feasibility constraints (2) at any time $t \in \mathbb{N}$; and we refer to them as trajectories. Clearly these constraints imply that $0 \leq z_t \leq 1$ for all $t \in \mathbb{N}$ so that $\mathbf{z}, \mathbf{u}, \mathbf{v}$ belong to $\ell_+^\infty = \ell_+^\infty(\mathbb{N})$.

Then, the optimization problem can be formulated as the search of optimal sequences $\mathbf{z}^*, \mathbf{u}^*, \mathbf{v}^*$ solution of the following mathematical program

$$P(z_0) \quad \begin{cases} V(z_0) = \max \sum_{t=0}^{\infty} b^t [U(u_t) + W(1 - z_t)] \\ \text{s.t. (2) and (3)} \\ \mathbf{z}, \mathbf{u}, \mathbf{v} \in \ell_+^\infty \text{ with } z_0 \text{ given.} \end{cases}$$

The existence of solutions for $P(z_0)$ follows as a consequence of the Weierstrass theorem since the feasible set is $\sigma(\ell^\infty, \ell^1)$ -compact and the objective function is $\sigma(\ell^\infty, \ell^1)$ -upper semi continuous. For details of the proof we refer to [3] where a similar problem without constraint (1) is treated. The proof remains valid with minor changes. We cannot assure the uniqueness of optimal solutions unless U and W are strictly concave, in which case we immediately get that \mathbf{u} and \mathbf{z} are unique, and uniqueness of \mathbf{v} rightly follows from the constraints.

The mathematical treatment of this problem is simpler if we state it only in terms of the state variable \mathbf{z} . As it is observed by McKenzie [10], it is a necessary condition of any optimal trajectory that production and consumption (i.e., determined by \mathbf{u} and \mathbf{v} in our model) are chosen in way so that the period's utility is maximized given the initial and terminal states. Hence, we perform an 'in period' maximization to find the optimal values of u_t and v_t as well as the maximum benefit of period t as a function of z_t and z_{t+1} . For $(z, z') \in [0, 1]^2$ we define

$$f(z, z') = \begin{cases} \max & U(u) + W(1 - z) \\ \text{s.t.} & 0 \leq u \leq z \\ & 0 \leq v \leq 1 - z \\ & z' = z - u + v \end{cases}$$

Let us first denote by $\mathcal{U}(z, z')$ the set of feasible controls that induce the transition of the system from state z to z'

$$\mathcal{U}(z, z') = \{(u, v) \in C(z) \mid z' = z - u + v\},$$

and notice that $\mathcal{U}(z, z')$ is never empty for any pair $(z, z') \in [0, 1]^2$. Then, the function $f(z, z')$ can be reformulated as

$$f(z, z') = \begin{cases} \max & U(u) + W(1 - z) \\ \text{s.t.} & (u, v) \in \mathcal{U}(z, z') \end{cases}$$

Looking carefully to the definition of the set \mathcal{U} we see that there is an even simpler expression

$$\mathcal{U}(z, z') = \{(u, v) \mid v = z' - z + u, u \in [\max(0, z - z'), \min(z, 1 - z')]\}$$

The maximum is attained when we take u equal to its upper bound. Within this approach, given a sequence of optimal states $\{z_t\}$, the optimal decisions can be calculated as

$$\begin{aligned} u_t &= \min(z_t, 1 - z_{t+1}), \\ v_t &= \min(1 - z_t, z_{t+1}). \end{aligned}$$

So, we get $f(z, z') = U(\min\{z, 1 - z'\}) + W(1 - z)$ and a dynamic optimization problem, equivalent to $P(z_0)$, can be written as

$$\tilde{P}(z_0) = \begin{cases} V(z_0) = \max \sum_{t=0}^{\infty} b^t f(z_t, z_{t+1}) \\ \text{s.t. } z_t \in [0, 1] \\ \text{and } z_0 \text{ given.} \end{cases}$$

The value function $V(z_0)$ associated with this optimal control problem can also be characterized by invoking the *dynamic programming principle*. It is a standard result (see for instance [1]) that V is the unique bounded function that satisfies Bellman equation

$$V(z) = \max_{z' \in [0, 1]} U(\min\{z, 1 - z'\}) + W(1 - z) + bV(z'). \quad (4)$$

Remark 1.1 Let $\mathcal{B}[V_0]$ be the Bellman operator defined on the set of bounded functions by

$$\mathcal{B}[V_0](z) = \max_{z' \in [0, 1]} U(\min\{z, 1 - z'\}) + W(1 - z) + bV_0(z').$$

The mapping \mathcal{B} is a contraction and the value function V is the unique fixed point with

$$V = \mathcal{B}[V] = \lim_{k \rightarrow \infty} \mathcal{B}^k[V_0]$$

for any bounded function V_0 (see [1] for details).

Lemma 1.2 The value function V is concave.

Proof. The Bellman operator preserves concavity. Consequently, when V_0 is concave the same holds for $\mathcal{B}[V_0]$, and consequently the value function V is a concave function. ■

2 The optimal trajectory

In this section we fully describe one solution of $P(z_0)$ for every $z_0 \in [0, 1]$. We emphasize that we do not attempt to describe completely the solution set of $P(z_0)$, but only to propose one optimal trajectory. In §2.1 we discuss two particular trajectories: one constant and the other periodic. They will turn out to be the asymptotic regime of the proposed optimal trajectory which becomes either constant or periodic after at most two time steps. In §2.2 we describe the transient stages before reaching the steady state or periodic regime. Finally, in §2.3 we find the value function and a feedback law that yields one optimal trajectory.

We will use the auxiliary C^1 concave function

$$\bar{U}(z) = U(z) + W(1 - z)$$

which represents the maximum benefit that one can get on a single time step when the resource is z .

2.1 Asymptotic behavior

In this subsection we look closely at two particular policies for problem $P(z_0)$: the *greedy periodic policy* and the *constant policy*, characterizing the states for which they are optimal.

Definition 2.1 (greedy periodic policy) *It consists in harvesting all the resource available and reallocating to the resource all the space that was assigned to the alternative use*

$$u_t = z_t, \quad v_t = 1 - z_t$$

generating a cyclic trajectory of period two: $z_{t+1} = 1 - z_t$.

The benefit of the periodic trajectory generated by the greedy policy from any initial condition $z \in [0, 1]$ is

$$J_G(z) = \frac{\bar{U}(z) + b\bar{U}(1-z)}{1-b^2}.$$

We define p to be the largest optimal solution of

$$(P_1) \quad \max_{z \in [0,1]} J_G(z)$$

Remark 2.2 *Directly from the definition of p and the concavity of J_G we know that*

- i. J_G is non-decreasing on $[0, p]$.
- ii. J_G is non-increasing on $[p, 1]$.

Denote $\Delta^p \subseteq [0, 1]$ the set of states whose value function is $V(z) = J_G(z)$, which may be characterized as follows.

Proposition 2.3 Δ^p is the set of points $z \in [0, 1]$ that satisfy

$$\bar{U}'(\max(1-z, z)) \geq b\bar{U}'(\min(1-z, z)). \quad (5)$$

Proof. Denote by $\phi(\cdot)$ the function

$$\phi(\xi, \xi') = U(\min(\xi, 1 - \xi')) + bJ_G(\xi').$$

If the greedy policy is optimal at z , then necessarily one has $V(z) = J_G(z)$ and $V(1-z) = J_G(1-z)$. From equation (4), this implies that the functions $\xi' \mapsto \phi(z, \xi')$ and $\xi' \mapsto \phi(1-z, \xi')$ are maximized respectively at $\xi' = 1-z$ and $\xi' = z$. Functions $\xi' \mapsto \phi(\xi, \xi')$ being concave (by composition of concave functions), this is also equivalent to require

$$0 \in \partial_2^+ \phi(\xi, 1 - \xi), \quad \xi \in \{z, 1 - z\}, \quad (6)$$

where $\partial_2^+ \phi$ denotes the Fréchet super-differential of ϕ w.r.t. to its second argument. Simple calculations give

$$\partial_2^+ \phi(\xi, 1 - \xi) = [-U'(\xi), 0] + bJ'_G(1 - \xi)$$

and

$$J'_G(1 - \xi) = \frac{\bar{U}'(1 - \xi) - b\bar{U}'(\xi)}{1 - b^2}.$$

Then, conditions (6) can be rewritten as

$$\begin{cases} \bar{U}'(1 - \xi) \geq b\bar{U}'(\xi) \\ U'(\xi) - b^2W'(1 - \xi) \geq b\bar{U}'(1 - \xi) \end{cases} \quad \xi \in \{z, 1 - z\},$$

Noticing that $U'(\xi) - b^2W'(1 - \xi) \geq \overline{U}'(\xi)$, these last conditions are equivalent to the simpler ones

$$\overline{U}'(1 - \xi) \geq b\overline{U}'(\xi), \quad \xi \in \{z, 1 - z\}.$$

Finally, $\overline{U}(\cdot)$ being concave, we obtain exactly the single condition (5).

Conversely, assume that condition (5) holds at a given z . Denote $p = \min(z, 1 - z)$ and notice that (5) is also fulfilled at any $z \in [p, 1 - p]$, because $\overline{U}(\cdot)$ is concave. Consider then the operator $\mathcal{M}_p : \mathcal{C}_{[0,1]} \rightarrow \mathcal{C}_{[0,1]}$ defined as follows

$$\mathcal{M}_p[f](\xi) = \begin{cases} f(1-p) + f'(1-p)(\xi - 1 + p), & \xi \in [0, 1-p] \\ f(\xi), & \xi \in [1-p, p] \\ f(p) + f'(p)(\xi - p), & \xi \in [p, 1] \end{cases}$$

and notice that $\mathcal{M}_p[f] \geq f$ for any $f \in \mathcal{C}_{[0,1]}$. Let \widetilde{V} be the value function for the modified utility functions $\widetilde{U} = \mathcal{M}_p[U]$, $\widetilde{W} = \mathcal{M}_p[W]$ and define the function $\widetilde{U}(z) = \widetilde{U}(z) + \widetilde{W}(1 - z)$. One has clearly $\widetilde{V} \geq V \geq J_G$. We claim that the function

$$\widetilde{J}_G(z) = \frac{\widetilde{U}(z) + b\widetilde{U}(1 - z)}{1 - b^2}$$

is solution of the Bellman equation (4). To see this first observe that the definition of \widetilde{U} implies that (5) is fulfilled for \widetilde{U} at any $z \in [0, 1]$, which in turn implies that \widetilde{J}_G is non-decreasing in $[0, 1]$. To see that

$$\widetilde{J}_G(z) = \widetilde{W}(1 - z) + \max_{z' \in [0,1]} \widetilde{U}(\min(z, 1 - z')) + b\widetilde{J}_G(z')$$

we distinguish two cases: (i) $z' \in [0, 1 - z]$, (ii) $z' \in [1 - z, 1]$.

$$\begin{aligned} (i) \quad \widetilde{W}(1 - z) + \max_{z' \in [0, 1 - z]} \widetilde{U}(\min(z, 1 - z')) + b\widetilde{J}_G(z') &= \widetilde{W}(1 - z) + \widetilde{U}(z) + \max_{z' \in [0, 1 - z]} b\widetilde{J}_G(z') \\ &= \widetilde{U}(z) + b\widetilde{J}_G(1 - z) = \widetilde{J}_G(z) \end{aligned}$$

$$\begin{aligned} (ii) \quad \widetilde{W}(1 - z) + \max_{z' \in [1 - z, 1]} \widetilde{U}(\min(z, 1 - z')) + b\widetilde{J}_G(z') &= \widetilde{W}(1 - z) + \max_{z' \in [1 - z, 1]} \widetilde{U}(1 - z') + b\widetilde{J}_G(z') \\ &= \widetilde{W}(1 - z) + \max_{z' \in [1 - z, 1]} -\widetilde{W}(z') + \widetilde{J}_G(1 - z') \\ &= \widetilde{J}_G(z) \end{aligned}$$

Consequently, one has $\widetilde{V}(z) = \widetilde{J}_G$. Finally, remark that $J_G = \widetilde{J}_G$ on $[1 - p, p]$, which proves that $V = J_G$ on $[1 - p, p]$. The greedy policy is then optimal from any $z \in [1 - p, p]$. \blacksquare

Notice that when $\overline{U}'(\frac{1}{2}) \geq 0$, we have $p \geq \frac{1}{2}$. As a consequence of Proposition 2.3 we deduce the following

Lemma 2.4 Δ^p is nonempty exactly when $\overline{U}'(\frac{1}{2}) \geq 0$, and then the set Δ^p is the interval $[1 - p, p]$. Furthermore, \overline{U} is non-decreasing on $[0, p]$.

Proof. Condition (5) of Proposition 2.3 and concavity of the function $\overline{U}(\cdot)$ imply the property

$$z \in \Delta^p \Rightarrow [\min(z, 1 - z), \max(z, 1 - z)] \subset \Delta^p \quad (7)$$

so Δ^p is an interval that necessarily contains $\frac{1}{2}$ whenever it is not empty. Now $\frac{1}{2}$ fulfills condition (5) exactly when $\overline{U}'(\frac{1}{2}) \geq 0$. The definition of p and the concavity of \overline{U} together with (7) imply $\Delta^p = [1 - p, p]$. The sign of $\overline{U}'(\frac{1}{2})$ and the concavity of \overline{U} imply that the function \overline{U} is non-decreasing on $[0, \frac{1}{2}]$ and from (5) this implies also that \overline{U} is non-decreasing on $[0, p]$. \blacksquare

Lemma 2.5 *If $\frac{1}{2} \leq p \leq 1$ then $V(z) \leq J_G(p)$ for all $z \in [0, 1]$.*

Proof. If $p = 1$ then the lemma above states that $V(z) = J_G(z)$ for all $z \in [0, 1]$ and the proof follows directly because $J_G(z)$ attains its maximum at $z = p$.

Let us consider now what happens when $p < 1$. In this case, the differentiable function $J_G(z)$ attains its maximum at an interior point of $[0, 1]$, thus $J'_G(p) = 0$. It is a known property that the slope of any concave function f , defined as: slope $f(z, z') = \frac{f(z) - f(z')}{z - z'}$, is non-increasing w.r.t. to its second variable (see for example [8]). As V is concave we have

$$\text{slope } V(x, p) \leq \text{slope } V(z, p) \quad \text{for all } x > p, z < p$$

From $V(z) \geq J_G(z)$ for all z and $V(p) = J_G(p)$ we easily deduce

$$\text{slope } V(z, p) \leq \text{slope } J_G(z, p) \quad \text{for all } z < p$$

Putting the two inequalities together we get

$$\text{slope } V(x, p) \leq \inf_{z < p} \text{slope } J_G(z, p) = 0 \quad \text{for all } x > p$$

where the equality follows from the fact that $J'_G(p) = 0$. And we easily conclude that $V(x) \leq V(p) = J_G(p)$ for $x < p$.

Analogously, we get $V(x) \leq V(p) = J_G(p)$ for $x > p$ and the lemma follows. ■

Remark 2.6 *Directly from the concavity of V and the fact that $V(z) \leq V(p)$ for all $z \in [0, 1]$ we know that*

- i. V is non-decreasing in $[0, p]$*
- ii. V is non-increasing in $[p, 1]$*

We will see in §2.3 that if $\bar{U}'(\frac{1}{2}) < 0$, the proposed optimal trajectory remains constant after at most two steps. This motivates the definition of

Definition 2.7 (constant policy) *It consists in harvesting and allocating always the quantity*

$$u_t = v_t = \min(z_t, 1 - z_t)$$

so that the trajectory is constant: $z_t = z_0$.

The benefit of the constant trajectory from the initial condition $z \in [0, 1]$ is given by

$$J_S(z) = \frac{U(\min(z, 1-z)) + W(1-z)}{1-b}.$$

Notice that $J_S(z) = \bar{U}(z)/(1-b)$ whenever $z \leq \frac{1}{2}$. This motivates the notion of a sustainable state, which corresponds intuitively to a state where it is optimal to stay forever.

Definition 2.8 (sustainable state) *A state $z \in [0, 1]$ is called sustainable if it is invariant under the optimal policy, i.e. the constant trajectory is optimal and $V(z) = J_S(z)$.*

Proposition 2.9 *The constant policy is optimal exactly at points z solutions of problem*

$$(P_2) \quad \max_{z \in [0, 1/2]} J_S(z)$$

We define \hat{z} to be the largest optimal solution of (P_2) . If $\hat{z} < 1/2$ then $V(z) \leq V(\hat{z}) = J_S(\hat{z})$ for all $z \in [0, 1]$.

Proof. Let us first show that the constant policy cannot be optimal for $z > \frac{1}{2}$. If $z > \frac{1}{2}$ then $\overline{U}(z), \overline{U}(1-z) > U(1-z) + W(1-z)$ and consequently

$$J_G(z) > \frac{(U(1-z) + W(1-z))(1+b)}{1-b^2} = J_S(z),$$

thus a contradiction with the optimality of the constant policy.

If the constant policy is optimal at $z \in [0, \frac{1}{2}]$, the following inequality is obtained from the Bellman equation (4)

$$\begin{aligned} V(z) = J_S(z) &\geq W(1-z) + U(\min(z, 1-\hat{z})) + bJ_S(\hat{z}) \\ &= U(z) + W(1-z) + bJ_S(\hat{z}) = J_S(z)(1-b) + bJ_S(\hat{z}) \end{aligned}$$

which implies $J_S(z) \geq J_S(\hat{z})$ i.e. $\overline{U}(z) \geq \overline{U}(\hat{z})$. So the constant policy cannot be optimal away from \hat{z} .

If $\hat{z} = \frac{1}{2}$, one has necessarily $\overline{U}'(\frac{1}{2}) \geq 0$ and from Lemma 2.4, the constant policy that coincides with the greedy policy at $\hat{z} = \frac{1}{2}$ is optimal. Otherwise, \hat{z} is a maximizer of the concave function \overline{U} on $[0, 1]$. From the Bellman equation (4), one can write the following inequalities

$$\begin{aligned} V(z) &= \overline{U}(z) + \max_{z' \in [0, 1]} U(\min(z, 1-z')) - U(z) + bV(z') \\ &\leq \overline{U}(\hat{z}) + b \max_{z' \in [0, 1]} V(z') = J_S(\hat{z})(1-b) + b \max_{z' \in [0, 1]} V(z') \end{aligned}$$

from which we deduce $V(\hat{z}) = J_S(\hat{z}) = \max_{z' \in [0, 1]} V(z')$. The constant policy is then optimal from \hat{z} . ■

Remark 2.10 *By the concavity of V and $V(z) \leq V(\hat{z})$ for all $z \in [0, 1]$ we immediately know that*

- i. V is non-decreasing in $[0, \hat{z}]$
- ii. V is non-increasing in $[\hat{z}, 1]$

2.2 Transient behavior

We claim that after two time steps the proposed optimal trajectory becomes either constant or periodic with period two, depending on the sign of $\overline{U}'(\frac{1}{2})$. Namely, if $\overline{U}'(\frac{1}{2}) < 0$ then after at most two steps the proposed optimal trajectory reaches \hat{z} and remains there afterwards. Otherwise, if $\overline{U}'(\frac{1}{2}) \geq 0$ it becomes 2-periodic after two time steps. More precisely, there are two different situations depending on the initial condition. If $z_0 \in [1-p, p]$ the optimal trajectory is a 2-periodic cycle from the beginning: $(z_0, 1-z_0, z_0, \dots)$. If $z_0 \notin [1-p, p]$, then the optimal trajectory reaches the $(p, 1-p)$ -cycle in one or two steps.

The previous subsection described the conditions under which the constant and greedy trajectories are optimal, from the first stage. In this subsection we discuss what happens when this is not the case, i.e., we describe precisely the behavior of the optimal trajectories during the two-step transient period before entering the periodic or steady-state regime.

We claim that the optimal policy during the transient is characterized by the auxiliary problem

$$Q(z_0) \quad \begin{cases} \max & U(u_0) + b\overline{U}(z_1) \\ \text{s.t.} & u_0 + z_1 \leq 1 \\ & 0 \leq u_0 \leq z_0 \\ & 0 \leq z_1 \leq p \end{cases}$$

This problem maximizes the two first steps benefit. The set of constraints is induced by (2) and (3) for stage $t=0$ except for $z_1 \leq p$. This last condition was introduced to assure that the two proposed asymptotic regimes are admissible after the transient. To see this, notice that $\hat{z} \leq p$ if and only if $\bar{U}'(\frac{1}{2}) \leq 0$.

For convenience we define the function

$$Q(z) = U(z) + b\bar{U}(1-z), \quad z \in [0, 1]$$

and denote by $q \in [0, 1]$ the point where it attains the maximum. If this point is not unique we take q as the largest optimal solution.

Problem $Q(z_0)$ can be solved explicitly: when $\bar{U}'(\frac{1}{2}) \leq 0$ one optimal solution is given by

$$(z_1, u_0) = \begin{cases} (\hat{z}, z_0) & \text{when } z_0 \in [0, 1-\hat{z}) \\ (1-z_0, z_0) & \text{when } z_0 \in [1-\hat{z}, q] \\ (1-q, q) & \text{when } z_0 \in (q, 1] \end{cases}$$

while if $\bar{U}'(\frac{1}{2}) > 0$ we have

$$(z_1, u_0) = \begin{cases} (p, z_0) & \text{when } z_0 \in [0, 1-p) \\ (1-z_0, z_0) & \text{when } z_0 \in [1-p, q] \\ (1-q, q) & \text{when } z_0 \in (q, 1] \end{cases}$$

2.3 The optimal policy

Using the results of the previous sections we may find explicitly the value function and the feedback law that gives the value z_{t+1} as a function of z_t for all t . We distinguish three different situations according to the position of the solution \hat{z} of (P_2) in the interval $[0, \frac{1}{2}]$: Left, Interior, or Right.

$$\begin{aligned} (\mathbf{S}_L) \quad & \hat{z} = 0, \quad \bar{U}'(0) \leq 0 \\ (\mathbf{S}_I) \quad & \hat{z} \in (0, \frac{1}{2}), \quad \bar{U}'(\hat{z}) = 0 \\ (\mathbf{S}_R) \quad & \hat{z} = \frac{1}{2}, \quad \bar{U}'(\frac{1}{2}) \geq 0 \end{aligned}$$

The graph of the feedback law is presented in Figure 1. To the left, we see the feedback law when (\mathbf{S}_I) holds

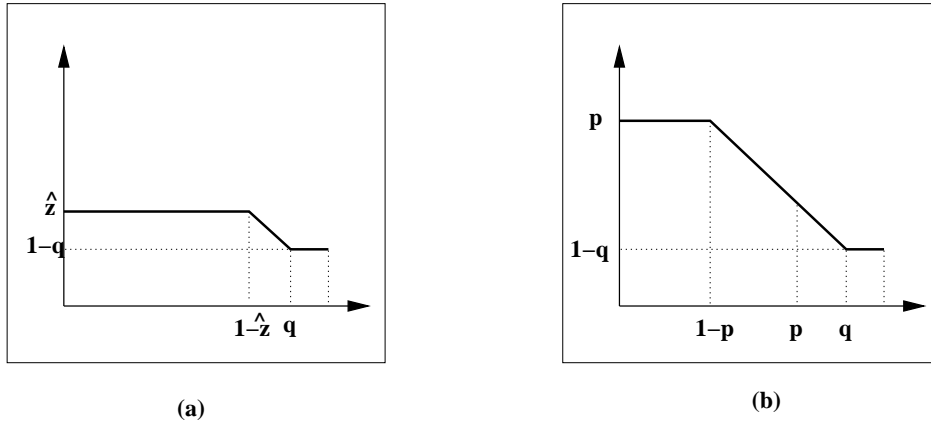


Figure 1: Feedback law.

and to the right when (\mathbf{S}_R) holds. The case (\mathbf{S}_L) can be seen as a degenerate case of the former where we simply have $z_t \equiv 0$ for all $t \geq 1$. In all situations it may happen that $q = 1$, in which case the last

interval disappears. The graphs are very similar and could be merged in a single diagram, but we prefer to distinguish them since they give rise to different dynamics.

In Figure 2 we present the optimal trajectory from a state $z_0 \in (p, q]$ when (\mathbf{S}_R) holds. Let us observe, that the trajectory reaches the periodic cycle $p, 1-p, p, \dots$ in two steps, the transient being $z_1 = 1 - z_0$. The evolution from any other initial state can be found similarly. We remark that for every initial state, the optimal trajectory reaches the $(p, 1-p)$ -cycle in one or two steps, except for the initial conditions in the interval $z \in [1-p, p]$, whose optimal trajectory is the periodic cycle $z, 1-z, z, \dots$ from the first stage.

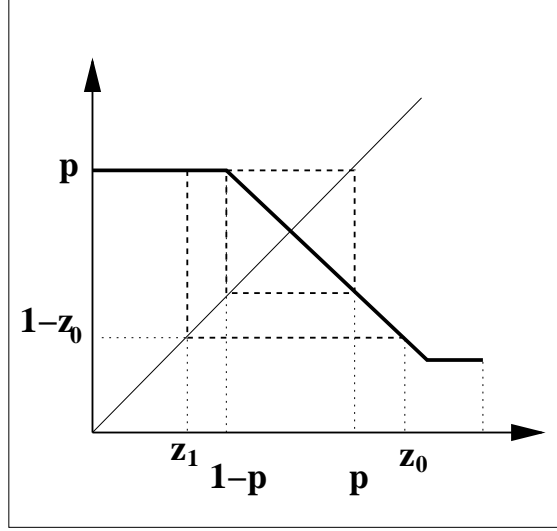


Figure 2: Dynamics

If (\mathbf{S}_I) holds the dynamics are even simpler and it is not difficult to see that there is convergence to the state \hat{z} in one or two steps.

The following theorem describes more precisely the feedback law represented in Figure 1 and proves that it is effectively optimal.

Theorem 2.11 *If (\mathbf{S}_L) or (\mathbf{S}_I) holds there is convergence in two steps to a sustainable state and the value function is*

$$V(z) = \begin{cases} \bar{U}(z) + bJ_S(\hat{z}), & z \in [0, 1 - \hat{z}] \\ W(1 - z) + \max_{z' \in [1 - \hat{z}, z]} Q(z') + b^2 J_S(\hat{z}), & z \in [1 - \hat{z}, 1] \end{cases}$$

which is attained by the optimal trajectory generated by the feedback law of Figure 1(a),

$$\begin{cases} (z_0, \hat{z}, \hat{z}, \hat{z}, \dots) & \text{when } z_0 \in [0, 1 - \hat{z}] \\ (z_0, 1 - z_0, \hat{z}, \hat{z}, \dots) & \text{when } z_0 \in (1 - \hat{z}, q] \\ (z_0, q, \hat{z}, \hat{z}, \dots) & \text{when } z_0 \in (q, 1] \end{cases} \quad (8)$$

Otherwise, if (\mathbf{S}_R) holds then Δ^p is reached in two steps and the optimal trajectory is greedy of period two afterwards, the value function being

$$V(z) = \begin{cases} \bar{U}(z) + bJ_G(p), & z \in [0, 1 - p] \\ J_G(z), & z \in [1 - p, p] \\ W(1 - z) + \max_{z' \in [p, z]} Q(z') + b^2 J_G(p), & z \in (p, 1] \end{cases}$$

which is attained by the optimal trajectory generated by the feedback law of Figure 1(b),

$$\begin{cases} (z_0, p, 1-p, p, 1-p, \dots) & \text{when } z_0 \in [0, 1-p) \\ (z_0, 1-z_0, z_0, 1-z_0, z_0, \dots) & \text{when } z_0 \in [1-p, p] \\ (z_0, 1-z_0, p, 1-p, p, \dots) & \text{when } z_0 \in (p, q] \\ (z_0, 1-q, p, 1-p, p, \dots) & \text{when } z_0 \in (q, 1] \end{cases} \quad (9)$$

Proof. Denoting

$$\Delta V(z, z') = W(1-z) + U(\min(z, 1-z')) + bV(z')$$

$V(\cdot)$ is the value function if it is solution of the Bellman equation (4), which amounts to prove that the equality

$$\max_{z' \in [0,1]} \Delta V(z, z') = V(z) \quad (10)$$

is fulfilled for any $z \in [0, 1]$.

Let us first study the case where (\mathbf{S}_R) holds. We distinguish three cases (i) $z \in [1-p, p]$, (ii) $z < 1-p$ and (iii) $z > p$

(i) It follows directly from Proposition 2.3 that $V(z) = J_G(z)$.

(ii)

$$\begin{aligned} \Delta V(z, z') &= W(1-z) + U(\min(z, 1-z')) + bV(z') \\ &\leq \bar{U}(z) + bV(z') \\ &\leq \bar{U}(z) + bJ_G(p) \end{aligned} \quad (\text{by Lemma 2.5})$$

Observing that $\Delta V(z, p) = \bar{U}(z) + bJ_G(p)$, it follows that $z' = p$ and $V(z) = \bar{U}(z) + bJ_G(p)$.

(iii) We start by using two properties of $\arg \max_{z' \in [0,1]} \Delta V(z, z')$.

1. If $z' \leq 1-z$, then $z' \leq 1-p$ and we have

$$\Delta V(z, z') = U(z) + W(1-z) + bV(z') \leq \bar{U}(z) + bV(1-z) \quad \text{for all } z' \leq 1-z$$

because $V(\cdot)$ is increasing in $[0, p]$ (see Remark 2.6). Hence $\arg \max \Delta V(z, z') \geq 1-z$.

2. If $z' \geq 1-p$, then $z' \geq p$ and we have

$$\Delta V(z, z') = U(1-z') + W(1-z) + bV(z') \leq W(1-z) + U(1-\hat{z}) + bV(\hat{z}) \quad \text{for all } z' \geq \hat{z}$$

because $V(\cdot)$ is decreasing in $[p, 1]$ (see Remark 2.6). Hence $\arg \max \Delta V(z, z') \leq p$.

This implies that $\max_{z' \in [0,1]} \Delta V(z, z') = \max_{z' \in [1-z, 1-p]} \Delta V(z, z')$. Finally

$$\begin{aligned} V(z) &= W(1-z) + \max_{z' \in [1-z, 1-p]} U(1-z') + bV(z') \\ &= W(1-z) + \max_{z' \in [1-z, 1-p]} U(1-z') + b[\bar{U}(z') + bJ_G(p)] \\ &= W(1-z) + \max_{z'' \in [p, z]} Q(z'') + b^2 J_G(p) \end{aligned}$$

We consider now what happens if (\mathbf{S}_I) or (\mathbf{S}_L) holds.

To prove that (10) is fulfilled for any $z \in [0, 1]$, we distinguish two cases: (i) $z \in [0, 1-\hat{z}]$ and (ii) $z > 1-\hat{z}$ observing that (i) comprises the whole interval whenever (\mathbf{S}_L) holds.

(i)

$$\begin{aligned}\Delta V(z, z') &= U(\min(z, 1 - z')) + W(1 - z) + bV(z') \\ &\leq U(z) + W(1 - z) + bJ_S(\hat{z}) \quad (\text{by Lemma 2.9})\end{aligned}$$

Observing that $\Delta V(z, \hat{z}) = \bar{U}(z) + bJ_S(\hat{z})$, it follows that $z' = \hat{z}$ and $V(z) = \bar{U}(z) + bJ_S(\hat{z})$.

(ii) By a process analogous to that of **(S_R.iii)** but using Remark 2.10 instead of Remark 2.6, it is not difficult to see that

$$\max_{z' \in [0, 1]} \Delta V(z, z') = \max_{z' \in [1-z, \hat{z}]} \Delta V(z, z')$$

and then

$$\begin{aligned}\max_{z' \in [1-z, \hat{z}]} \Delta V(z, z') &= W(1 - z) + \max_{z' \in [1-z, \hat{z}]} U(\min(z, 1 - z')) + bV(z') \\ &= W(1 - z) + \max_{z' \in [1-z, \hat{z}]} U(1 - z') + b(\bar{U}(z') + bJ_S(\hat{z})) \quad (\text{by } \mathbf{(S_I.i)}) \\ &= W(1 - z) + \max_{z' \in [1-\hat{z}, z]} Q(z') + b^2 J_S(\hat{z})\end{aligned}$$

■

Remark 2.12 *If the delay before reallocation of the primary use was not compulsory, that is to say if we remove constraint (1), then the reserved space can be considered as allocated to another virtual species with a maturity age equal to one. Then, the problem is easily solved and it can be shown that an optimal trajectory converges to a new sustainable state \tilde{z} , given by $\tilde{z} \in \arg \max_{z \in [0, 1]} \bar{U}(z)$ in only one time step, the optimal trajectory being constant afterwards. In this case, the sustainable state is the same as the one of our model when condition **(S_L)** or **(S_I)** holds. The qualitative behavior of both problems is the same with convergence to the same state. However, under condition **(S_R)**, the situation is different. In this case, if reserving the space was not mandatory then the sustainable state would change: we would have $\tilde{z} > 1 - \tilde{z}$. Furthermore, in the problem without constraint (1), there might be convergence to a sustainable state with $\tilde{z} = 1$, that is to say the alternative use is not present at all. This represents a situation where the alternative use does not provide enough benefit to endure by itself.*

3 Conclusions

We have analyzed a discrete time model for the optimal harvesting of a renewable resource, where the space occupancy is limited and reallocation after harvesting is not immediately allowed (assuming that the maturity age of the resource is reached after one time step).

Techniques based on dynamic programming and concavity has led us to characterize the optimality of two myopic strategies leading to constant or greedy periodic trajectories, and to make explicit the optimal trajectory from any possible initial state (along with the optimal transient decisions to be kept).

This paper shows that considering an alternative use of the resource appears to be different to the abstract problem simply adding a virtual species, because of the delay imposed before reallocating the space to its primary use. This leads to a non-trivial optimal policy, for which a different anticipation scheme might be required, before using a myopic decision.

The assumption of a net benefit of the alternative use depending only on the amount cleared by the harvest, and separable from the net revenue of the harvest could be considered to be oversimplified for real world economics. Nevertheless, this has allowed us to provide a complete rigorous solution of the problem, and to develop a methodology based on the combination of dynamic programming and concavity tools, that could be re-used for more complicated models. This is first step toward more realistic modeling.

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A Mathematical programming proofs

A.1 Asymptotic behavior

Throughout we let \mathbf{z} be an optimal trajectory for $P(z_0)$.

To avoid working with the non-differentiable function $\min(\cdot, \cdot)$ we will sometimes use the following equivalent formulation of $P(z_0)$

$$\begin{cases} V(z_0) = \max & \sum_{t=0}^{\infty} b^t [U(u_t) + W(1 - z_t)] \\ \text{s.t.} & 0 \leq u_t \leq z_t \\ & u_t + z_{t+1} \leq 1 \\ & \text{and } z_0 \text{ given.} \end{cases} \quad (11)$$

Proof. (*Proposition 2.3*) To prove the necessity of condition (5) take $z_0 \in \Delta^p$ and consider the greedy trajectory issued from z_0 : $\mathbf{z} = (z_0, 1 - z_0, z_0, 1 - z_0, \dots)$. Suppose that $1 - z_0 > z_0$ and consider the perturbed trajectory $\mathbf{z}_\epsilon = (z_0, 1 - z_0 - \epsilon, z_0 + \epsilon, 1 - z_0 - \epsilon, \dots)$ where $\epsilon > 0$.¹ If the original trajectory is optimal, the benefit obtained with it (V) must be greater or equal to the one obtained with the new trajectory (V_ϵ), this is

$$\begin{aligned} V - V_\epsilon &= \frac{b}{1-b^2} [U(1 - z_0) - U(1 - z_0 - \epsilon)] + \frac{b^2}{1-b^2} [U(z_0) - U(z_0 + \epsilon)] \\ &\quad + \frac{b}{1-b^2} [W(z_0) - W(z_0 + \epsilon)] + \frac{b^2}{1-b^2} [W(1 - z_0) - W(1 - z_0 - \epsilon)] \geq 0. \end{aligned}$$

Dividing by $\epsilon > 0$ and letting it to 0 yields (5). The proof in the case $z_0 > 1 - z_0$ is completely analogous, perturbing the optimal trajectory at $t = 1$ instead of $t = 0$.

To prove the sufficiency of condition (5) let us show that the greedy periodic trajectory is optimal for $P(z_0)$. We prove the optimality using the Karush-Kuhn-Tucker (KKT) theorem. To this end we consider the Lagrangian associated with the optimization problem stated in (11)

$$\mathcal{L} = \sum_{t=0}^{\infty} b^t [U(u_t) + W(1 - z_t) + \mu_t u_t + \alpha_t (z_t - u_t) + \beta_t (1 - z_{t+1} - u_t)] \quad (12)$$

together with the following set of ℓ^1 -multipliers

$$\begin{cases} \mu_t = 0 \\ \alpha_t = \frac{b^t}{1-b^2} [Q'(z_t) - b^2 W'(1 - z_t)] \\ \beta_t = \frac{b^{t+1}}{1-b^2} [\bar{U}'(1 - z_t) - b \bar{U}'(z_t)]. \end{cases} \quad (13)$$

In fact, the multipliers belong to ℓ_+^1 due to condition (5). We can see that we have complementary slackness and the feasibility constraints are fulfilled. To prove that $\nabla \mathcal{L} = 0$, it is left to see that $\mathcal{L}_{u_t} = \mathcal{L}_{z_t} = 0$ for all t .

¹This feasible perturbed trajectory is obtained by allocating $1 - z_0 - \epsilon$ at $t = 0$ instead of $1 - z_0$, keeping this extra space in the alternative use for one period and allocating $z_0 + \epsilon$ at $t = 1$, after which we continue with a greedy periodic policy.

$$\begin{aligned}
\mathcal{L}_{u_t} &= b^t U'(u_t) + \mu_t - \alpha_t - \beta_t \\
&= b^t U'(z_t) - \frac{b^t}{1-b^2} [Q'(z_t) - b^2 W'(1-z_t)] - \frac{b^{t+1}}{1-b^2} [\bar{U}'(1-z_t) - b \bar{U}'(z_t)] \\
&= \frac{b^t}{1-b^2} [(1-b^2)U'(z_t) - U'(z_t) + b \bar{U}'(1-z_t) + b^2 W'(1-z_t) - b \bar{U}'(1-z_t) + b^2 \bar{U}'(z_t)] = 0
\end{aligned} \tag{14}$$

$$\begin{aligned}
\mathcal{L}_{z_t} &= -b^t W'(1-z_t) + \alpha_t - \beta_{t-1} \\
&= -b^t W'(1-z_t) + \frac{b^t}{1-b^2} [Q'(z_t) - b^2 W'(1-z_t)] - \frac{b^t}{1-b^2} [\bar{U}'(1-z_{t-1}) - b \bar{U}'(z_{t-1})] \\
&= \frac{b^t}{1-b^2} [(b^2-1)W'(1-z_t) + U'(z_t) - b \bar{U}'(1-z_t) - b^2 W'(1-z_t) - \bar{U}'(z_t) + b \bar{U}'(1-z_t)] = 0
\end{aligned} \tag{15}$$

Thus, the proposed trajectory is a stationary point of the Lagrangian (12) and hence a solution of $P(z_0)$. \blacksquare

Proof. (Proposition 2.9). Let us show that the stationary trajectory $z_t = \hat{z}$ is optimal for $P(\hat{z})$. To this end consider the Lagrangian (12) with the following set of ℓ_+^1 -multipliers

$$\begin{cases} \mu_t = \beta_t = 0 \\ \alpha_t = b^t U'(\hat{z}) \end{cases}$$

A routine verification shows that $\nabla \mathcal{L} = 0$ and we have complementary slackness which implies that the proposed trajectory is a stationary point for $P(\hat{z})$. Hence, it is an optimal solution and \hat{z} is sustainable.

To prove the other implication, let \tilde{z} be a sustainable state. We claim that $\tilde{z} \in \arg \max_{z \in [0, \frac{1}{2}]} \bar{U}(z)$. To this end, consider the following trajectory $\tilde{z} = \tilde{z} - \epsilon \quad \forall t \geq 1$. Given that \tilde{z} is a sustainable state, the benefit obtained with the constant trajectory must be greater or equal to the one obtained with the alternative trajectory, which is equivalent to

$$U(\tilde{z}) - U(\tilde{z} - \epsilon) + W(1 - \tilde{z}) - W(1 - \tilde{z} + \epsilon) \geq 0 \tag{16}$$

If $\tilde{z} = \frac{1}{2}$, then the proposed alternative trajectory is feasible for $0 < \epsilon < \frac{1}{2}$. So dividing (16) by ϵ and letting $\epsilon \rightarrow 0$ yields

$$U'(\tilde{z}) - W'(1 - \tilde{z}) \geq 0 \iff \bar{U}'(\tilde{z}) \geq 0$$

which readily implies that $\tilde{z} = \frac{1}{2} \in \arg \max_{z \in [0, \frac{1}{2}]} \bar{U}(z)$. If $\tilde{z} \in (0, \frac{1}{2})$ we can consider ϵ positive and negative provided that $|\epsilon|$ is small enough. So, repeating the reasoning we get

$$\bar{U}'(\tilde{z}) = 0$$

which is equivalent to $\tilde{z} \in \arg \max_{z \in [0, \frac{1}{2}]} \bar{U}(z)$. And finally, if $\tilde{z} = 0$, only the values of $\epsilon < 0$ can be considered and this yields $\bar{U}'(0) \leq 0$, which also gives us $\tilde{z} \in \arg \max_{z \in [0, \frac{1}{2}]} \bar{U}(z)$. \blacksquare

A.2 Optimal policy

Proof. (Theorem 2.11). If (\mathbf{S}_L) holds we propose the primal solution $z_t = 0$ for all $t \geq 1$ with the ℓ_+^1 -multipliers $\mu_t = \beta_t = 0$ and $\alpha_t = b^t U'(z_t)$. It is easy to see that the proposed trajectory is a stationary

point of the Lagrangian (12) and thus an optimal solution to $P(z_0)$.

If (\mathbf{S}_I) holds, z_t evolves following the feedback law showed in Figure 1(a), reaching \hat{z} in at most two steps. We recall the proposed optimal trajectory

$$(8) \quad \begin{cases} (z_0, \hat{z}, \hat{z}, \hat{z}, \dots) & \text{when } z_0 \in [0, 1-\hat{z}] \\ (z_0, 1-z_0, \hat{z}, \hat{z}, \dots) & \text{when } z_0 \in (1-\hat{z}, q] \\ (z_0, q, \hat{z}, \hat{z}, \dots) & \text{when } z_0 \in (q, 1] \end{cases}$$

We propose the following ℓ^1 -multipliers:

$$\begin{aligned} \mu_t &= 0 \\ \alpha_t &= \begin{cases} U'(u_0) - b\bar{U}'(z_1) & t = 0 \\ bU'(z_1) & t = 1 \\ b^t U'(\hat{z}) & t \geq 2 \end{cases} \\ \beta_t &= \begin{cases} b\bar{U}'(z_1) & t = 0 \\ 0 & t \geq 1 \end{cases} \end{aligned}$$

It is evident that $\alpha_t \geq 0$ and $\beta_t \geq 0$ for all $t \geq 1$. To see the non-negativity of α_0 and β_0 , we calculate them as a function of z_0

$$\begin{aligned} \alpha_0 &= \begin{cases} U'(u_0) & \text{if } z_0 \leq 1-\hat{z} \\ Q'(z_0) & \text{if } 1-\hat{z} < z_0 \leq q \\ 0 & \text{if } z_0 > q \end{cases} \quad (Q'(z_0) \geq Q'(q) = 0) \\ \beta_0 &= \begin{cases} 0 & \text{if } z_0 \leq 1-\hat{z} \\ b\bar{U}'(1-z_0) & \text{if } 1-\hat{z} < z_0 \leq q \\ b\bar{U}'(1-q) & \text{if } z_0 > q \end{cases} \quad \begin{aligned} &(\bar{U}'(1-z_0) \geq \bar{U}'(\hat{z}) = 0) \\ &(\bar{U}'(1-q) \geq \bar{U}'(\hat{z}) = 0) \end{aligned} \end{aligned}$$

We proceed now to prove that $\nabla \mathcal{L} = 0$,

$$(14) \implies \begin{aligned} \mathcal{L}_{u_0} &= U'(u_0) - U'(u_0) + b\bar{U}'(z_1) - b\bar{U}'(z_1) = 0 \\ \mathcal{L}_{u_1} &= bU'(z_1) - bU'(z_1) - 0 = 0 \\ \mathcal{L}_{u_t} &= b^t U'(\hat{z}) - b^t U'(\hat{z}) - 0 = 0 \end{aligned} \quad \text{for all } t \geq 2$$

$$(15) \implies \begin{aligned} \mathcal{L}_{z_1} &= -bW'(1-z_1) + bU'(z_1) - b\bar{U}'(z_1) = 0 \\ \mathcal{L}_{z_t} &= -b^t W'(1-\hat{z}) + b^t U'(\hat{z}) = 0 \end{aligned} \quad \text{for all } t \geq 2$$

We turn now to case (\mathbf{S}_R) . Here, z_t evolves following the feedback law showed in Figure 1(b), becoming periodic in at most two steps. We recall here the proposed optimal trajectory

$$(9) \quad \begin{cases} (z_0, p, 1-p, p, 1-p, \dots) & \text{when } z_0 \in [0, 1-p] \\ (z_0, 1-z_0, z_0, 1-z_0, z_0, \dots) & \text{when } z_0 \in [1-p, p] \\ (z_0, 1-z_0, p, 1-p, p, \dots) & \text{when } z_0 \in (p, q] \\ (z_0, 1-q, p, 1-p, p, \dots) & \text{when } z_0 \in (q, 1] \end{cases}$$

We observe that case $z_0 \in [1-p, p]$ has been already studied in Proposition 2.3. To see the optimality of

these trajectories we propose the following ℓ_+^1 -multipliers: $\mu_t = 0$ and

$$\alpha_t = \begin{cases} U'(u_0) - b\bar{U}'(z_1) + \frac{b^2}{1-b^2}[\bar{U}'(z_2) - b\bar{U}'(1-z_2)] & t = 0 \\ b\bar{U}'(z_1) - \frac{b^2}{1-b^2}[\bar{U}'(z_2) - b\bar{U}'(1-z_2)] & t = 1 \\ \frac{b^t}{1-b^2}[\bar{U}'(z_t) - b\bar{U}'(1-z_t)] + b^t W'(1-z_t) & t \geq 2 \end{cases}$$

$$\beta_t = \begin{cases} b\bar{U}'(z_1) - \frac{b^2}{1-b^2}[\bar{U}'(z_2) - b\bar{U}'(1-z_2)] & t = 0 \\ \frac{b^{t+1}}{1-b^2}[\bar{U}'(1-z_t) - b\bar{U}'(z_t)] & t \geq 1 \end{cases}$$

The definition of the multipliers is quite involved but it is possible to find simpler expressions if we divide the study into four cases depending on the initial state z_0 .

$$z_0 \in [0, 1-p) \implies \alpha_t = \begin{cases} U'(z_0) & t = 0 \\ b^t W'(1-p) & t \neq \dot{2} \\ b^t U'(1-p) & t = \dot{2} \end{cases} \quad \beta_t = \begin{cases} b^t U'(p) & t \neq \dot{2} \\ 0 & t = \dot{2} \end{cases}$$

$$z_0 \in [1-p, p] \implies \text{Idem (13)}$$

$$z_0 \in (p, q] \implies \alpha_t = \begin{cases} Q'(z_0) & t = 0 \\ b\bar{U}'(1-z_0) & t = 1 \\ b^t U'(1-p) & t \neq \dot{2}, t \geq 2 \\ b^t W'(1-p) & t = \dot{2}, t \geq 2 \end{cases} \quad \beta_t = \begin{cases} b\bar{U}'(1-z_0) & t = 0 \\ b^t \bar{U}'(p) & t = \dot{2}, t \geq 2 \\ 0 & t \neq \dot{2} \end{cases}$$

$$z_0 \in (q, 1] \implies \alpha_t = \begin{cases} 0 & t = 0 \\ U'(q) & t = 1 \\ b^t U'(1-p) & t \neq \dot{2}, t \geq 2 \\ b^t W'(1-p) & t = \dot{2}, t \geq 2 \end{cases} \quad \beta_t = \begin{cases} U'(q) & t = 0 \\ b^t \bar{U}'(p) & t = \dot{2}, t \geq 2 \\ 0 & t \neq \dot{2} \end{cases}$$

Following arguments already seen in this proof and after a long but straightforward computation, we see that the proposed trajectory is a stationary point of \mathcal{L} and hence, it is an optimal solution to $P(z_0)$. Finally, it is easy to see that the discounted total benefit given by the optimal trajectory is effectively the proposed value function, which proves the theorem. \blacksquare

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