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MODELOS MATEMÁTICOS PARA LA GESTIÓN ÓPTIMA DE RECURSOS NATURALES RENOVABLES

UNA APLICACIÓN A LA GESTIÓN SUSTENTABLE DE UNA ZONA FORESTAL MIXTA

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA MENCIÓN MODELACIÓN MATEMÁTICA

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MODELOS MATEMÁTICOS PARA LA GESTIÓN ÓPTIMA DE RECURSOS NATURALES RENOVABLES

Una aplicación a la gestión sustentable de una zona forestal mixta

La orientación general de este trabajo concierne la explotación óptima de recursos naturales renovables con un interés particular en las plantaciones forestales compuestas por múltiples especies.

Estudiamos, desde el punto de vista del control óptimo en tiempo discreto, el comportamiento asintótico de una plantación forestal donde las especies tienen diferentes velocidades de crecimiento y diferentes funciones de beneficio asociadas. Se prueba la existencia y la unicidad de un estado homogéneo e invariante bajo la trayectoria óptima, conocido como el *estado sustentable.* Se presentan condiciones bajo las cuales toda trayectoria óptima converge asintóticamente al estado sustentable. Según nuestro conocimiento, ésta es la única prueba del hecho que el estado sustentable es un atractor global. Se prueba que en el caso general, toda trayectoria óptima se acerca al conjunto de trayectorias periódicas óptimas.

La caracterización del comportamiento asintótico permite abordar nuevos problemas. En primer término, consideramos la inclusión en el modelo de un mercado de la tierra, dando la posibilidad de negociar una fracción de la superficie en cada período de tiempo. Posteriormente adaptamos el modelo, prohibiendo la siembra inmediata después de la cosecha. Esta hipótesis podría corresponder a restricciones de orden reglamentario o técnico. Se compara entonces una situación donde el descanso de la tierra después de la cosecha es obligatorio con otra donde el mismo es opcional.

Finalmente, intentamos relacionar las dos comunidades científicas que estudian la explotación de las plantaciones forestales: la economía forestal (a la cual pertenecen todos los resultados mencionados hasta ahora) y la ingeniería forestal. Nuestro objetivo es construir modelos, siempre dentro del marco del control óptimo, que tengan en cuenta algunas características consideradas imprescindibles por los forestales, como el diámetro promedio y la altura máxima, para luego iniciar el análisis matemático de estos modelos.

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"If the solution is to be simple,
the assumptions must be heroic."
Paul A. Samuelson

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Introduction

This thesis concerns the study of optimal management of renewable resources with a particular focus on multiple species forests.

The forest exploitation to get economic benefit dates from many centuries and the first ideas to make this benefit maximal appeared as early as the 15^{th} century. However, there are not many works applying mathematical modelling to this problem as compared to other renewable resources such as fishes in the sea. Works related to mixed forests are even scarcer. The growth of an uncontrolled forest is still not completely understood and models are generally species and site-specific, mostly oriented to numerical simulations. Thus, finding the optimal management policy is a largely open mathematical problem.

The thesis is divided in two parts.

The first part comprising chapters 1, 2, 3 and 4, deals with the *characterization of the asymptotic behavior of an optimally managed multiple species forest*. We treat the forest management problem using models and techniques coming from forest economics. We make the very strong yet widely used assumption that the value of a tree is determined only by its age, i.e., the forest's growth is considered as a pure aging process.

In Chapter 1, we give a brief historical review of forest economics, discussing the most important mathematical results, starting from the groundbreaking paper of M. Faustmann.

Chapter 2 studies the asymptotic behavior of the optimal harvesting policies for a mixed forest with multiple species having different maturity ages. We prove the existence and uniqueness of an homogeneous state invariant under the optimal policy, the so-called *sustainable state*. We discuss the conditions under which an optimal trajectory converges in the long run towards this state or towards an optimal periodic cycle.

In the following chapter we replicate this study when the forest is inserted in a land market and land trading is possible at every moment. We prove the existence of *sustainable states* in this new context, that are no longer unique, which makes the study technically more difficult. We discuss as well conditions under which there is asymptotic convergence towards one of these states or towards an optimal periodic cycle.

In Chapter 4, we study the optimal harvesting of a renewable resource that cannot be continuously exploited, i.e., after the harvest the released space cannot be immediately reallocated to its main use, but in the meantime, it can be given an alternative use with positive utility. Although similar, this model behaves quite differently to the previous one, and its study is much more involved.

In the second part of this thesis, corresponding to chapters 5 and 6, we attempt to make the link between two scientific communities that study the forest from different points of view: the forest economics and the forestry. All the models presented in the first part, correspond to forest economics. Our objective is to study and develop forest growth models taking into account more than simply the age of the forest and suitable to the mathematical analysis of the free and controlled behavior of the forest.

Even when neglecting the value that a standing tree may have due to its numerous environmental benefits, and only focusing on the economic benefit that can be obtained when it is harvested, we should know the quantity and the quality of the timber a tree contains to estimate properly its economic value. Estimating the quality is very difficult and depends strongly on the possible lumber's uses, specially since different parts of a tree may have different uses. We will concentrate in forest growth models, without taking into account the quality issue.

To approximate accurately the volume of a tree, forest engineers use generally its height and its diameter at breast height. There are several factors whose effect upon height and diameter may be considered unavoidable by forest researchers depending on the particular situation. To mention some of them: climate; water availability; soil composition and slope; competition with other trees for light, water and nutrients; animals living in the forest; etc.

Numerous forest growth models have been developed with various levels of detail, and with an emphasis on either mechanistic process representation or on accurate long-term forecasting. The most appropriate model types depend on the intended use, forest characteristics, and other circumstances. In Chapter 5 we present a classification of growth models following [28], together with a brief bibliographical review focused on models suitable to tackle management issues.

In the spirit of finding a theoretical tree value function that takes into account some characteristics of the forests other than age, we study in detail two forest growth models. In Chapter 6, we work with a stand model presented by Garcia [12] and we present the continuous version of the Maugé's model. Both of them were developed for pine stands. We find explicitly the trajectory they describe when no interventions are made, we study its optimal periodic exploitation and we present extensions to a two species forest.

In the remainder of this introduction we provide a synthesis of the results obtained in this thesis. In each section we present the main problem we have considered, we give the necessary definitions that help to understand the subject, and we describe our contribution to the subject. For a complete description, we refer the reader to the following chapters.

Part I: Modelling the forest growth as a pure aging process

Chapter 1: A discussion of infinite-horizon optimization models in Forest Economics

We give in this chapter a brief historical review of the mathematics of the forest economics to help to understand the relative importance of the problems studied in this thesis. We discuss and compare the known results, focusing on infinite horizon problems with no a-priori assumptions about the asymptotic behavior of the optimal policies.

In 1849, Martin Faustmann correctly specified the problem of finding the economic value of an even-aged forest stand [13]. Considering an infinite horizon discrete time model and periodic policies, he obtained an expression for the present value of the stand, a question solved by Ohlin in 1921 who characterized the optimal rotation period, now known as the *Faustmann harvesting age* [26]. The generalization of the optimal rotation problem to a forest with many even-aged stands was already considered at Faustmann's time, but its complete resolution remains open even today.

Historically, it was assumed that any optimal managed forest would converge towards a very particular homogeneous state, the so-called *sustainable state* (some authors call it *normal forest* or *fully regulated forest*). In this state, the surface of the forest is evenly distributed among even-aged stands or "age classes", whose ages range from 1 to the Faustmann harvesting age.

It was necessary to wait until 1985, to have the harvesting problem with multiple age classes specified in a form fruitful for mathematical analysis in Mitra and Wan papers [22, 23]. They studied a forest divided in age classes which is annually harvested. One of their main results is the invariance of the sustainable state under the optimal policy, but they also show that the popular belief about the long-run behavior of the forest state is not true: in general there is no convergence to such a state.

The articles by Salo and Tahvonen [31]-[33] improve these results using mathematical programming techniques. Also, Rapaport, Sraidi and Terreaux [30] considered a single species forest where harvest is forbidden before the maturity age and such that a tree's value remains constant after this age, showing that every optimal trajectory becomes periodic after a finite time. In particular this implies that the problem can be reduced to a finite dimensional one and solved numerically.

Salo and Tahvonen treated the case of a single species forest combined with an alternative annual use of the land [31], a case that can be interpreted as a special two species forest. Aiming to simplify the dynamics, the harvesting policy is predetermined: the oldest age-class and the alternative use are clear-cut, nothing else is harvested. This leaves only one decision per stage: the allocation of the cleared land between the two possible uses. In this setting, they prove that at least one sustainable state exists and that it is a local saddle point, presenting some numerical examples that show convergence of the optimal trajectory to such a state.

Chapter 2: Asymptotic convergence of optimal harvesting policies for a multiple species forest

This chapter presents the complete characterization of the asymptotic behavior of a multiple species forest, a problem that had been open since Faustmann times.

We consider the extension of the Rapaport-Sraidi-Terreaux model to the case of a mixed forest composed by several species with different maturity ages. We generalize the results of Salo and Tahvonen [33] for a one species forest and a competing annual use of the land, by considering multiple species with arbitrary maturity ages and using a different methodology based on Lyapunov functions that allow to establish global convergence results. The main advantage of this approach is that proofs do not depend on the number of species or age classes, while the method of proof provided in [33] is not possible to generalize.

We establish that the sustainable state is a *global attractor* whenever the maturity ages of the species present at this state are relatively prime, and that there is asymptotic convergence towards the set of *greedy periodic cycles* in the general case.

We consider a discrete time model for the optimal management of a forest of total area S occupied by k species $I = \{1, \ldots, k\}$ with maturity ages of $n_1, \ldots, n_k$ years respectively. For each period $t \in I\!N$ we denote $x_t^i \geq 0$ the area of species $i \in I$ that reaches its maturity in year t, and $\bar{x}_t^i \geq 0$ the area occupied by over-mature trees (older than n_i). A convenient representation of the forest in terms of the age distribution at time t is provided by the *state* $\mathbb{X}_t = (X_t^1, \ldots, X_t^k)$ where $X_t^i = (x_{t+n_i-1}^i, x_{t+n_i-2}^i, \ldots, x_t^i, \bar{x}_t^i)$ describes the areas occupied in year t by trees of species i with ages $1, 2, \ldots, n_i$ and over n_i.

We must decide how much land $u_t^i \geq 0$ to harvest and how to reallocate this land to new seedlings. Assuming that only mature trees can be harvested we must have $u_t^i \leq \bar{x}_t^i + x_t^i$, and then the area not harvested in that period will comprise the over-mature trees at the next step, namely

(1) $$\bar{x}^i_{t+1} = \bar{x}^i_t + x^i_t - u^i_t.$$

The total harvested area $\sum_{i\in I} u^i_t$ is allocated to new seedlings that will reach maturity in years $t + n_i$ respectively, which is expressed by the equation

(2) $$\sum_{i\in I} x^i_{t+n_i} \;=\; \sum_{i\in I} u^i_t.$$

The total benefit obtained from the harvests is given by the value $V = \sum_{i\in I}\sum_{t=0}^{\infty} b^t U_i(u^i_t)$ where $b \in (0,1)$ is a discount rate and $U_i : I\!R \to I\!R$ is smooth, increasing and strictly concave for each $i \in I$. We denote $\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i$ the sequences of states and controls. Since all the areas are non-negative and smaller than S these sequences belong to the ℓ^∞ ball B^∞_S of radius S and centered at the origin. The state evolution consists of an age-shift dynamics, except for the first and last components of each vector X^i_t which are controlled by the sowing and harvesting policies. If $\mathbb{X}_0$ corresponds to the initial state reflecting the age-class composition of the forest at time $t = 0$, the problem to be solved may be stated as

(3) $$P(\mathbb{X}_0)\;\begin{cases}\displaystyle\max_{\boldsymbol{x}^i,\bar{\boldsymbol{x}}^i,\boldsymbol{u}^i}\text{imize}\;\; V = \sum_{i\in I}\sum_{t=0}^{\infty} b^t U_i(u^i_t)\\ \text{subject to (1) and (2)}\\ \boldsymbol{x}^i,\bar{\boldsymbol{x}}^i,\boldsymbol{u}^i \in \ell^\infty_+ \text{ with } \mathbb{X}_0 \text{ given.}\end{cases}$$

The existence of solution is consequence of the Weierstrass theorem because the feasible set is non empty and $\sigma(\ell^\infty,\ell^1)$-compact while the objective function is $\sigma(\ell^\infty,\ell^1)$-upper semi continuous. However, only the uniqueness of the $\boldsymbol{u}^i$ can be assured.

DEFINITION. *A state is called* sustainable *if it is invariant under an optimal policy.*

We prove that the sustainable state always exists and is unique. Even more, it is of the form $X^{*i} = (x^{*i},\ldots,x^{*i},0)$ where x^* is the solution to the strictly concave problem

(4) $$(S)\;\begin{cases}\displaystyle\max_{x\in I\!R^k}\text{imize}\;\; \sum_{i\in I} n_i\sigma_i U_i(x^i)\\ \text{subject to}\;\; x^i \geq 0 \text{ and } \sum_{i\in I} n_i x^i = S.\end{cases}$$

where $\sigma_i = \frac{b^{n_i}}{1-b^{n_i}}$. Intuitively σ_i is the discounted value of an infinite sequence of planting cycles with harvesting age n_i and unitary benefit, that is, $\sigma_i = b^{n_i} + b^{2n_i} + \ldots$.

DEFINITION. *A feasible trajectory is called* greedy *if every tree is harvested as soon as it reaches maturity ($\bar{x}^i_t = 0$). It will be called a* greedy periodic cycle (GPC) *if in addition $x^i_{t+n_i} = u^i_t$, i.e., the harvested area is sown with seedling of the same species it had before the harvest.*

We characterize completely the set Δ^p of initial states for which the corresponding optimal trajectory is a GPC in terms of the discount rate and the benefit functions. Even more, we prove that the set of GPCs comprises only the sustainable state whenever the maturity ages of the species present at it are relatively prime. All the results concerning the sustainable state and the GPCs are proved exploiting the mathematical programming theory along with convexity and duality.

Let $\Phi : \Delta \rightarrow I\!R$ be given by

$$\Phi(\mathbb{X}_0) = G(\mathbb{X}_0) - \sum_{i \in I} \sum_{t=0}^{n_i - 2} \frac{b^t - b^{n_i - 1}}{1 - b^{n_i}} U_i(x_t^i)$$

where $G(\mathbb{X}_0)$ is the optimal benefit obtained from state $\mathbb{X}_0$ by using a greedy policy.

We show that, if the optimal trajectory is greedy, Φ is a Lyapunov function modulo N where N is the *least common multiple* of the n_i. Even more, $\Phi(\mathbb{X}_{t+N}) > \Phi(\mathbb{X}_t)$ whenever $\mathbb{X}_t \notin \Delta^p$, and using this we are able to determine the convergence of every optimal trajectory to the set of GPCs. It is worth mentioning that even if Φ is a Lyapunov function only when restricted to greedy trajectories, the result is true in general, and we have the main result of this article, stated as Theorem 2.4.8

THEOREM. *Every optimal trajectory of $P(\mathbb{X}_0)$ converges to a GPC in the sense that*

$$\lim_{t \to \infty} \text{dist}(\mathbb{X}_t, \Delta^p) = 0.$$

In particular if $\Delta^p = \{\mathbb{X}^*\}$, as is the case when the maturity ages of the species present at the sustainable state are relatively prime, then $\lim_{t \to \infty} \mathbb{X}_t = \mathbb{X}^*$.

Finally, we identify different situations under which the convergence occurs in finite time. When this happens $P(\mathbb{X}_0)$ can be reformulated as a finite dimensional problem and solved numerically. Unfortunately, this is not true in general even for states which admit a greedy optimal trajectory. We restrict the analysis to a case with two species since the general case is too involved, and achieve the explicit characterization of the initial conditions whose optimal trajectory do converge after finitely many steps.

Chapter 3: Land Market

This chapter deals with an extension of the previous model to include a land market. Here, we allow any fraction of the land to be sold or bought at every time step. We model the price as a function both of the land traded and the stock of land available in the market.

Let S denote the total surface, $a_t = \sum_{i \in I} \left[\bar{x}_t^i + \sum_{j=0}^{n_i - 1} x_{t+j}^i \right]$ the area occupied at time t and c_t the fraction of land traded at time t (we take $c_t > 0$ if some land is bought and $c_t < 0$ if it is sold). We choose the function $W(a, c) = -\int_a^{a+c} \rho(\xi) d\xi - \gamma |c|$ to model the cost of the land transaction, where $\rho(\cdot) : [0, S] \rightarrow I\!R_+$ is a continuous, non-decreasing price function which takes into account that the scarcer the land is, the more expensive it becomes, while the term $-\gamma|c|$ incorporates transaction costs such as administrative expenses.

The optimization problem is now stated as (compare with Problem $P(\mathbb{X}_0)$ defined in (3))

$$P(\mathbb{X}_0)\left\{\begin{array}{ll}\underset{\boldsymbol{x}^i,\bar{\boldsymbol{x}}^i,\boldsymbol{u}^i,\boldsymbol{a},\boldsymbol{c}}{\text{maximize}} & V=\sum_{t=0}^{\infty} b^t\big[\sum_{i\in I} U_i(u_t^i)+W(a_t,c_t)\big]\\ \text{subject to} & a_{t+1}=a_t+c_t, \qquad 0\le a_t\le S.\\ & \bar{x}_{t+1}^i=\bar{x}_t^i+x_t^i-u_t^i.\\ & \sum_{i\in I} x_{t+n_i}^i=c_t+\sum_{i\in I} u_t^i.\\ & \boldsymbol{x}^i,\bar{\boldsymbol{x}}^i,\boldsymbol{u}^i,\boldsymbol{a}\in\ell_+^\infty,\ \boldsymbol{c}\in\ell^\infty \text{ with } \mathbb{X}_0 \text{ given.}\end{array}\right.$$

To see even clearer the similarity between these two problems, notice that the objective function can be expressed as

$$\begin{aligned} V &= \textstyle\sum_t b^t\big[\sum_{i=1}^k U_i(u_t^i)-\int_0^{a_{t+1}}\rho(\xi)d\xi+\int_0^{a_t}\rho(\xi)d\xi-\gamma|c_t|\big]\\ &= \textstyle\frac{1}{b}\int_0^{a_0}\rho(\xi)d\xi+\sum_t b^t\big[\sum_{i=1}^k U_i(u_t^i)-\frac{1-b}{b}\int_0^{a_t}\rho(\xi)d\xi-\gamma|c_t|\big]\end{aligned}$$

where the first term depends only on the initial conditions. This suggests to consider the unused land, $x_t^0=S-a_t$, as a new species X^0 with benefit function $U_0(x^0)=-\frac{1-b}{b}\int_0^{S-x^0}\rho(\xi)d\xi$ and maturity age $n_0=1$. Notice that $U_0(\cdot)$ is smooth, increasing and concave as a function of x^0 and that $c_t=x_t^0-x_{t+1}^0$. Hence, we may restate the problem as

$$P(\mathbb{X}_0)\left\{\begin{array}{ll}\underset{\boldsymbol{x}^i,\bar{\boldsymbol{x}}^i,\boldsymbol{u}^i,\boldsymbol{c}}{\text{maximize}} & \sum_{t=0}^{\infty} b^t\big[\sum_{i=0}^k U_i(u_t^i)-\gamma|c_t|\big]\\ \text{subject to} & \bar{x}_{t+1}^i=\bar{x}_t^i+x_t^i-u_t^i \qquad i\in\{0,\cdots,k\}\\ & \sum_{i=0}^k x_{t+n_i}^i=\sum_{i=0}^k u_t^i\\ & x_{t+1}^0=x_t^0-c_t\\ & \boldsymbol{x}^i,\bar{\boldsymbol{x}}^i,\boldsymbol{u}^i\in\ell_+^\infty,\boldsymbol{c}\in\ell^\infty \text{ with } \mathbb{X}_0 \text{ given.}\end{array}\right.$$

From the point of view of the mathematical technique, the main difficulty is that the term $\gamma|\cdot|$ is non-differentiable. Due to this, the uniqueness of the sustainable state is lost and new optimal periodic cycles appear, making the analysis more difficult (uniqueness is only recovered if $\gamma=0$).

Each surface a supports at most one associated sustainable state, which is an homogeneous state as before: $X_a^i=(x_a^{*i},\dots,x_a^{*i},0)$, where the vector $x_a^*=(x_a^{*1},\dots,x_a^{*k})$ is the unique solution of Problem (S_a) defined as in (4) but replacing the area constraint with $\sum_{i\in I} n_i x_a^i=a$. We call r_a its Lagrange multiplier. However not every surface a yields a sustainable state. We prove that such surfaces comprise an interval $[\underline{a},\overline{a}]\subseteq[0,S]$, where the values of $\underline{a}$ and $\overline{a}$ are uniquely determined by γ, $\rho(\cdot)$ and r_a. Even more, only the surfaces in $[\underline{a},\overline{a}]$ may admit an optimal GPC.

Even if the characterization of the stationary trajectories is much more involved, the main result is still valid: we have convergence to a sustainable state or to the set of optimal periodic cycles, however, we cannot determine a priori the final occupied surface.

We are able to prove this convergence, not only if U_0 is strictly concave, as would be the analogous to the problem studied in Chapter 2, but also when U_0 is linear, i.e., when the forest owner is not capable of affecting the land price in the market. Besides, the same result is also true when the function U_0 is simply concave, provided that the optimal trajectory becomes greedy after finitely many steps.

In all these situations, we prove that land trading converges asymptotically to zero. In the particular case where U_0 is linear, this trading is bounded in time. If in addition, $\gamma = 0$, then the optimal trajectory can be explicitly found and it reaches the unique sustainable state in finite time.

Finally, we briefly discuss the asymptotic behavior of a forest where land conversion between species is costly. We see that this problem is a generalization of the main topic of this article and that the same type of asymptotic behavior is expected. This issue was briefly discussed by Salo and Tahvonen in [33] for the case of a two species forest, when one of them is annual. They present numerical examples where new optimal periodic cycles appear when conversion costs are introduced.

Chapter 4: Optimal control of renewable resources with alternative use

Sometimes there may exist legal or technical issues that prevent the land from being immediately resowed after the harvest. This situation arises from practical constraints such as land refreshing, cleaning, or other resting duties that do not allow an immediate re-use of the space for the forest. We assume in this study that the harvested space has to wait for at least one period before becoming available to be reallocated again to the resource. In the meantime, the space can be given an alternative use with positive utility.

The constraint of no immediate reallocation, makes this model different from the one of a unique species forest with an alternative annual use, and gives rise to a different behavior of optimal policies.

We denote by U and W the benefit function associated respectively to the forestry and the alternative use, as before they are assumed to be smooth, increasing and strictly concave. We represent the state of the forest by $\mathbb{X}_t = \big((x_{t+n-1}, \ldots, x_t, \bar{x}_t), y_t\big)$ where n is the maturity age of the forest species, $x_t \geq 0$ is the area of trees reaching its maturity in year t and $y_t \geq 0$ is the area allocated to the alternative use in year t. The problem to be solved may be stated as

$$P(\mathbb{X}_0)\left\{\begin{array}{ll}\underset{\boldsymbol{x},\bar{\boldsymbol{x}},\boldsymbol{y},\boldsymbol{u}}{\text{maximize}} & \sum_{t=0}^{\infty} b^t[U(u_t)+W(y_t)] \\ \text{subject to} & \bar{x}_{t+1}=\bar{x}_t+x_t-u_t \\ & x_{t+n}+y_{t+1}=u_t+y_t \\ & x_{t+n}\le y_t \\ & \boldsymbol{x},\bar{\boldsymbol{x}},\boldsymbol{y},\boldsymbol{u}\in\ell_+^\infty \text{ with } \mathbb{X}_0 \text{ given.}\end{array}\right.$$

We prove that the sustainable state is a point of the form $\mathbb{X}^* = \big((x^*,\dots,x^*,0),y^*\big)$ where x^* and y^* are solution of the following optimization problem

$$(S)\left\{\begin{array}{ll}\underset{x,y\,\in\,I\!R}{\text{maximize}} & n\,\sigma_n U(x)+\sigma_1 W(y) \\ \text{subject to} & 0\le x\le y \\ & nx+y=S\end{array}\right.$$

The solution falls naturally into three mutually exclusive cases

($\mathbf{S_L}$) If $\sigma_n U'(0)\le\sigma_1 W'(S)$ then $x^*=0$ and $y^*=S$

($\mathbf{S_I}$) If $\sigma_n U'(x^*)=\sigma_1 W'(y^*)$ with $y^*>x^*>0$ and $nx^*+y^*=S$

($\mathbf{S_R}$) If $\sigma_n U'(\frac{S}{n+1})\ge\sigma_1 W'(\frac{S}{n+1})$ then $x^*=y^*=\frac{S}{n+1}$

We also characterize the greedy periodic cycles, whose definition is different from the one used in the previous chapters,

DEFINITION. *A* greedy periodic cycle *consists in harvesting all the available resource and reallocating to the resource all the space that was previously assigned to the alternative use*

Notice, that such a cycle is $(n+1)$-periodic, instead of n-periodic as would be the case of a two species forest where one of them is annual.

Treating this constraint together with the delay inherent to the forest growth, is technically very difficult in the general case. So, in a first approach we assume that the resource is regenerated in only one time step. Of course, this is not suitable to model a forest, hence, in the first part we prefer to talk of an abstract *renewable resource*.

We solve the system completely, finding explicitly the optimal trajectory of the system for every time step. Two different approaches are used to establish the results: first an approach based on dynamic programming and then mathematical programming theory is exploited, along with convexity and duality.

In this particular case where seedlings can be harvested at the end of the time period, there is no need to distinguish between age classes, hence we call $z_t\ge 0$ the fraction of space occupied by the resource[1]. The new formulation of the problem is more suitable to use dynamic programming tools than the former.

As before $u_t \geq 0$ denotes the harvested land and we introduce v_t the fraction of the land occupied by the alternative use that will be reallocated to the resource, $0 \leq v_t \leq 1 - z_t$.

The optimization problem to be solved can be expressed as

$$P(z_0) \begin{cases} \underset{\boldsymbol{u}, \boldsymbol{v}}{\text{maximize}} & \sum_{t=0}^{\infty} b^t [U(u_t) + W(1 - z_t)] \\ \text{subject to} & z_{t+1} = z_t - u_t + v_t \\ & 0 \leq u_t \leq z_t \\ & 0 \leq v_t \leq 1 - z_t \\ & \boldsymbol{u}, \boldsymbol{v} \in \ell_+^\infty \text{ with } z_0 \text{ given.} \end{cases}$$

The value function $V(z_0)$ associated with this optimization problem can also be characterized by invoking the *dynamic programming principle*. It is a standard result (see for instance [4]) that V is the unique bounded function that satisfies Bellman equation, which after some manipulations can be expressed as

$$V(z) = W(1-z) + \max_{z' \in [0,1]} U(\min(z, 1 - z')) + bV(z').$$

Within this approach, an optimal feedback law $z_{t+1} = f(z_t)$ can be deduced. In Figure 1(a)

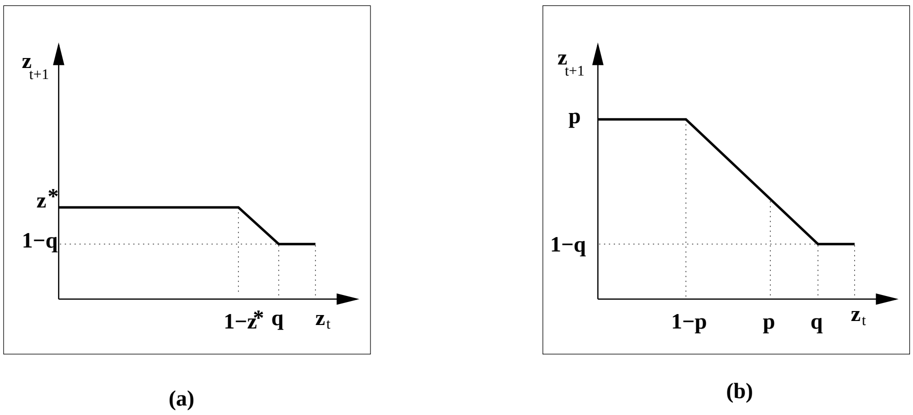

Figure 1: Feedback law.

we see the feedback law when $(\mathbf{S_L})$ or $(\mathbf{S_I})$ holds and in (b) when $(\mathbf{S_R})$ holds. The critical values p and q are characterized as solutions to very simple, well-defined optimization problems. In both situations it may happen that $q = 1$, in which case the last interval disappears. The graphs

[1] $z_t = \bar{x}_t + x_t$

are very similar and could be merged in a single diagram, but we prefer to distinguish them since they give rise to different dynamics. It is easy to see from these feedback laws that every optimal trajectory reaches either z^* the sustainable state, or a 2-periodic cycle in at most two steps.

Conclusions of this first part are dressed together with a comparison of the advantages of each one of the approaches.

In the second part of this chapter, we study a case where the regeneration delay is arbitrary, presenting some partial results. We use only the mathematical programming approach, because the greater number of variables makes more difficult to apply dynamic programming. In addition to the already seen characterization of the stationary trajectories, i.e., the sustainable state and the greedy periodic cycles, we are able to prove that every optimal trajectory becomes greedy after a bounded interval of time. More importantly, we fully determine the asymptotic behavior of the system, whenever the utility function W is linear.

Part II: Other characteristics of the forest growth process

Chapter 5: A bibliographical review

We start the second part of the thesis presenting the systematic classification by Porté and Bartelink of the numerous forest growth models. This three level classification places the emphasis in models suitable to mixed forest.

The first classification criteria used by Porté and Bartelink is the smallest unit identified, distinguishing in the first place between *tree-level*, also called individual models that describe and keep track of each individual tree on the stand, and *stand-level* which consider the forest as a unit and describe the stand by a small number of aggregate "macro" variables, such as mean diameter, top height, trees per hectare, etc. The second criteria is spatial dependence, i.e., both tree-level and stand-level models can be divided into distance dependent and distance independent models. The third criterion describes whether or not forest heterogeneity is taken into account. A sketch of the classification is shown in Figure 2.

We then proceed with a brief review of the literature concerning forestry models. We focus on articles where stand-level models for mixed forests are designed and some management issues are discussed. Although extremely short, the review includes very diverse examples. In particular, we mention the following articles

1. Stand-level	→	(a) Distance dependent		
	→	(b) Distance independent	→	i. Average tree models
			→	ii. Distribution models
2. Tree-level	→	(a) Distance dependent		
	→	(b) Distance independent	→	i. Gap models
			→	ii. Non gap models

Figure 2: Sketch of Porté and Bartelink classification.

(Boungiorno et al. [6]) a discrete-time three-species matrix model of stand growth is proposed an validated with data from the mixed Jura mountain forest. Four different management policies are evaluated with ecological and economical criteria. It is worth mentioning that the silvicultural interventions are allowed only at some predetermined periodic points in time.

(Jögiste [14]) a discrete-time average tree model of stand growth is designed. Three populations are distinguished. Simulations are presented and some general remarks about the optimal management policy are deduced from them.

(Crépin [11]) a continuous-time nonlinear model of the boreal forest ecosystem is presented. The model comprises two forest species and one herbivore species that interact between themselves. Some necessary conditions of the time-continuous optimal management rules are deduced, although the existence of such optimal policies is not assured.

(Clutter [8]) a continuous time, “compatible” model is stated. In this more than forty years old article Clutter expressed for the first time the existing relationship between growth and yield, defining the so-called compatible models. We include it because it introduced this important concept and also because we will consider in Chapter 6 a model based on it. The author compares two different silvicultural programs comprising interventions at discrete points.

(Ochi & Cao [25]) In this article there is no discussion about management. The authors make a comparison between two “Clutter’s type" compatible models and one annual recursive new model, designed by them. We comment it briefly, taking the opportunity to present more examples of growth and yield stand models.

Chapter 6: Clutter-Garcia model and Maugé model

We would like to state the optimal harvesting problem in a way suitable to make the link between the characteristics of the growth process and the management decisions. In the first part of the thesis, the forest growth is considered as a pure aging process but this is a very strong assumption and part of the scientific community disagree with it, considering that many other factors affect the trees growing and the timber quality in a non-negligible way.

With the objective of developing a theoretical tree value function that takes into account some characteristics of the forests other than age, we study in detail two forest growth models. Our objective is to characterize the optimal harvesting policy in this framework. We work with relative simple stand models to allow mathematical tractability.

Firstly, we study a model for a single-species, even-aged loblolly pine stand presented by Garcia [12] based on previous work of Clutter [8] and Clutter and Lenhart [10]. The state variables of this model are the top height and the basal area of the stand, which may be enough to make good estimations of the volume of timber contained in the forest. It consists of a system of differential equations that describe the evolution of a stand left by itself. Garcia assumes that the silvicultural treatments occur at discrete points, causing an immediate change of state.

We start by solving explicitly the set of differential equations. Then, we apply the simplest possible harvesting policy: the Faustmann harvesting, i.e., the whole stand is periodically cut down. We study different timber volume estimation functions found in the literature based on the variables modelled: top height and basal area. Assuming that the timber price is linear and with or without considering the planting cost, the existence and uniqueness of the optimal harvesting period is proved. An implicit characterization of such optimal period is given.

Finally, we develop an extension to two species of the Clutter-Garcia model, where space competition is considered. The new model is not explicitly solvable in general, but it is possible to characterize its asymptotic behavior when it evolves freely, which differs of the single species one.

We then proceed with the study of a model presented by Maugé in 1975, this is a time discrete stand model, that was initially calibrated for maritime pine and has been adapted afterwards for other types of pines. The variables are top height of the stand and the average circumference.

We study its continuous version, solving the resulting system of differential equations. The study of the previous section is repeated: we prove state the problem of finding the optimal harvesting period, proving its existence and uniqueness for two different stand value functions.

This chapter is a first approach to the linkage between the two scientific communities that study the forests. It aims at briefly presenting some of the difficulties encountered when working with models that reflect partially the forest complexity. Even if the two chosen models are relatively simple, the optimization problem becomes much more difficult to deal with and only partial results are found in the particular case of periodic exploitation.

Part I

Modelling the forest growth as a pure aging process

Chapter 1

A discussion of infinite-horizon optimization models in Forest Economics

1.1 Even-aged forest: Faustmann

In 1849, Martin Faustmann published his famous article "Calculation of the value which forest land and immature stands possess for forestry" [13] where he presents a general theory of forests' valuation. Thanks to this article he is considered today as the beginner of the modern forestry economics. Apparently he was the first to see the relationship between the value of a forest and its optimal management. He states that the value of a forest is related to the benefits it can provide presently and in the future and hence, in choosing the optimal management we are maximizing this value.

Considering an infinite horizon discrete time model and periodic policies, he obtained an expression for the present value of an even-aged forest, formula that can be used to determine the optimal harvesting age. Stating the problem in infinite horizon, Faustmann takes into account that the longer the cut of the existing forest is delayed the longer it takes to obtain benefit from future harvests. The optimization problem was solved by Ohlin in 1921 proving the existence of an optimal rotation period, the so-called *Faustmann age*, and characterizing it [26]. The elegance and simplicity of Ohlin's result stems from the fact that he dealt with forests of identically aged trees.

Faustmann, as well as most of the forest economics authors, considers that the volume of timber contained in a forest depends only on its age s, representing this volume by the function $f(s)$. This is, of course, a very strong assumption as there are many other factors that influence the trees growing that foresters consider unavoidable. In the second part of this thesis we will study two forest growth models, in a first approach to build f functions considering some other characteristics of the forests.

Another important hypothesis of the Faustmann model is that the price p per unit of timber volume is known and constant. Even more, this price is not influenced by the age of the harvested trees, as it could be the case given that timber quality changes during a tree lifetime.

Finally, the discount rate $b \in (0,1)$ is also considered constant in time. Thus, the present value of a forest harvested at time t, when trees are s years old is exactly $b^t pf(s)$.

Observe that the restriction to periodic policies made by Faustmann was not really necessary, given that the periodicity of the optimal harvesting policy is assured by the invariance with respect to time of the price and the biomass function [7].

Hence, assuming that at time $t = 0$ the tract of land is empty, the Faustmann problem is stated as:

$$\max_s \sum_{t \in I\!N} b^{st} pf(s) = \frac{b^s}{1-b^s} pf(s) \tag{1.1}$$

Clearly the optimal harvesting period is any m that satisfies:

$$\frac{b^m}{1-b^m} f(m) \geq \frac{b^s}{1-b^s} f(s) \quad \text{for all } s. \tag{1.2}$$

A priori, this period is not unique. As in the previous chapter, we denote $\sigma_m = \frac{b^m}{1-b^m}$.

The previous question can also be stated in continuous time, as some authors do, but we prefer to work in discrete time. In forestry this choice is largely justified since the periods of growing and harvesting are delimited through the year.

1.2 Forest with multiple even-aged stands

The generalization of the optimal rotation problem to a forest with many even-aged stands was already considered at Faustmann's time, but its complete resolution remains open even today. Nevertheless, Faustmann's ideas were extremely influential and inspired various harvesting rules which present a long run behavior that guarantees a sustainable and regular flow of timber.

Today, the forest economics literature proposes many different ways of finding the optimal harvesting policy. Solutions range from simple heuristics, simulations, linear optimizations models, optimal control problems, exhaustive search of solutions, etc. But most of these works assume a priori that the desirable long-run state of the forest's population is some even land allocation between the even-aged stands (also called *age classes*) or include different types of even-flow constraints.

A natural long run equilibrium is the so-called *normal forest* or *sustainable state* in which the land is evenly allocated among all the existing age classes, with a rotation period equal to a Faustmann age. This is to say that if m is solution to (1.2) and the total area is S, each of the stands with ages from $1, \dots, m$ occupy $\frac{S}{m}$ of the land and there are no trees older than m.

Faustmann himself presents in this article a method to go from bare land to the normal forest, but it is not optimal. Indeed, the rule that Faustmann proposes, is only empirical and its economic effectiveness remains to be proved. It is the only part of the article which is not completely rigorous and clearly justified.

Only recently, with Mitra and Wan's articles [22, 23], the forestry age class problem has been stated in a way useful for the economic analysis. Mitra and Wan examine this problem from an analytical point of view and obtain results that challenge some historical paradigms, providing examples where the optimal trajectory consists of a periodic cycle which *does not* converge to the normal forest. In the following we cite some of the main results of these authors together with some results of Salo and Tahvonen and Rapaport, Sraidi and Terreaux.

1.2.1 Mitra and Wan

In [22, 23] Mitra and Wan studied the problem faced by a forest manager who wishes to maximize the total benefit he or she can get when such benefit depends only on the timber obtained. The forest is divided in different age classes and is annually harvested. They propose a discrete time optimization problem whose objective function is the sum of the actualized benefits over an infinite horizon.

Inside every age class all the trees are considered identical and the timber content of each area unit depends only on the trees age through the biomass function f. The economic benefit depends on the total timber volume and is given by the function U. The following assumptions concerning the function f and the benefit function U are stated in their work

H1 $f(s) = 0$ for $0 \leq s \leq \mathrm{s}$, for some $\mathrm{s} \geq 1$.

H2 f is continuous, and there is an integer $n > \mathrm{s}$ such that

(i) $f(s)$ is increasing for $s < n$ and

(ii) $f(s)$ is decreasing for $s > n$.

H3 f is concave for $s \geq \mathrm{s}$.

H4 U strictly increasing.

H5 U continuous in $\mathbb{R}_+$ and $U \in C^2(\mathbb{R}_{++})$.

H6 U concave.

The discount rate is denoted by $b \in (0,1)$ and without loss of generality it is assumed that the total area S is equal to 1. It is also assumed that new seedlings are planted immediately in the cleared areas.

The state of the forest at the end of the time period t is represented by the vector $X_t = (x_t^1, \ldots, x_t^n)$, where x_t^s is the area occupied by trees of age s at time t.

Mitra and Wan claimed in their article of 1985: "[...] Also, for any reasonable objective function for the economy, trees will never be allowed to grow beyond age n; we therefore take this as a condition of feasibility itself". This reasoning allows the authors to take an n-dimensional state vector. We make the observation that the fact that trees are not allowed to grow beyond n when an optimal policy is applied, can not be assured only with hypothesis H1-H6. In fact, the concavity of the function U favors homogeneous harvests at each stage, hence it may be convenient to postpone the harvesting beyond age n in order to reshape the forest into a more homogeneous state.

This is circumvented very easily by taking, instead of an n-dimensional state vector, a N-dimensional one, where N is the age at which a tree dies.

Nevertheless, we present here the problem with an n-dimensional state vector, as Mitra and Wan proposed it. However, we make an important change: Mitra and Wan represent the state of the forest at the beginning and at the end of every time period, while the only process between this two variables is the growing of the trees, i.e., a one position shift to the right. We discard this differentiation representing the forest only at the end of the time period just before the silvicultural processes take place. This make the notation lighter and coherent with the one used in the rest of this thesis. We also denote $f(s)$ by f_s to make the notation more compact and easier to handle.

At every time step we must decide how much land of every age class z_t^s to harvest, and then the total harvested volume is $c_t = \sum_{s=1}^{n} f_s z_t^s$. Due to the extra hypothesis we already know that $z_t^n = x_t^n$. The trees of the age class s, not harvested in one period will comprise the age class $(s+1)$ at the following step, i.e., $x_{t+1}^{s+1} = x_t^s - z_t^s$ for all $s = 1, \ldots, n-1$ while x_{t+1}^1 is determined through the area constraint $\sum_{s=1}^{n} x_{t+1}^s = 1$.

The total benefit obtained from the harvests is given by the value $V = \sum_{t=0}^{\infty} b^t U(c_t)$. The problem is to find the sequence of harvests $z_t^s \geq 0$ which maximizes the value of V while keeping the state variables x_t^s non-negative subject to the constraints above.

We denote $\boldsymbol{x}^s, \boldsymbol{z}^s$ and $\boldsymbol{c}$ the sequences of states, controls and total harvested volumes. Since all the areas are non-negative and smaller than 1 all these sequences belong to ℓ_+^∞.

Given the initial state X_0 at time $t=0$, the optimization problem may be stated as

$$(1.3) \qquad (\mathrm{P}_{X_0}) \begin{cases} \underset{\boldsymbol{z}^s, \boldsymbol{x}^s, \boldsymbol{c}}{\text{maximize}} & \sum_{t \in I\!N} b^t U(c_t) \\ \text{s.a.} & c_t = \sum_{s=1}^n f_s z_t^s \\ & z_t^n = x_t^n \\ & x_{t+1}^{s+1} = x_t^s - z_t^s \qquad s = 1, \ldots, n-1 \\ & \sum_{s=1}^n x_t^s = 1 \\ & x_0^s = X_0^s \end{cases}$$

The original Faustmann problem is being generalized in many ways. Firstly, the benefit function is concave and not simply linear. This could reflect the problem of a forest manager capable of modifying the market price. Secondly, the problem is stated with an arbitrary initial forest distribution, and more importantly, they allow the harvesting of any fraction of any age class while Faustmann only allows the falling of the whole even-aged forest at a certain time.

Nothing is commented here about the existence of an optimal solution for $P(X_0)$. It can be proved, for example, from Theorem 4.6 in Stokey and Lucas [36], where they established the equivalence of this problem with the solution of the corresponding Bellman's equation if the benefit function is bounded. This is indeed the case, given that U is continuous and $\boldsymbol{c} \in \ell^\infty$.

In Chapter 2 we provide a proof of the existence of solution to a similar optimization problem based on the Weierstrass theorem, that can be easily adapted to this case.

Eliminating variables c_t y z_t^s the problem may be written more simply as

$$(\mathrm{P}_{X_o}) \begin{cases} \underset{\boldsymbol{x}^s}{\text{maximize}} & \sum_{t \in I\!N} b^t \left[U(\sum_{s=1}^{n-1} f_s(x_t^s - x_{t+1}^{s+1}) + f_n x_t^n) \right] \\ \text{s.a.} & \sum_{s=1}^n x_t^s = 1 \\ & x_t^s - x_{t+1}^{s+1} \geq 0 \qquad \forall s = 1, \ldots, n-1 \\ & x_0^s = X_0^s \end{cases}$$

Optimal stationary trajectory

Let m be a solution to the Faustmann problem (1.2). In the sequel, we denote by X^* the state

$$(1.4) \qquad X^* = \big(\underbrace{\tfrac{1}{m}, \ldots, \tfrac{1}{m}}_{m}, \underbrace{0, \ldots, 0}_{n-m} \big)$$

that corresponds to the normal forest. Mitra and Wan show that this state is invariant under the optimal policy.

[22, THEOREM 3.1] *If H1,H2,H4-H6 hold, X^* is invariant under the optimal policy.*

Observe that the harvesting policy that leaves X^* invariant consists in clear-cutting the oldest age class, leaving the rest untouched and resowing immediately the liberated land. At the following step, when every tree is one year older, we have exactly the same state X^*.

To prove the uniqueness of the invariant state an extra hypothesis is needed:

H7 There is a unique solution m to the Faustmann problem (1.2).

[22, THEOREM 5.1] *Under H1-H7, X^* is the unique invariant state.*

The authors also prove that if the uniqueness condition is not satisfied, H3 and the convexity of the σ_s's with respect to s, imply that the maximum of (1.2) is attained in two consecutive values m and $m+1$. More recently, Salo and Tahvonen [32] prove that in this case, there exists a continuum of invariant states of the form

$$X^* = \big(\underbrace{x, \ldots, x}_{m}, y, \underbrace{0, \ldots, 0}_{n-m-1}\big) \qquad \text{where } mx + y = 1 \text{ and } y \in [0, \tfrac{1}{m+1}]$$

We include now, our own proofs of Theorems 3.1 and 5.1 of [22] based in mathematical programming and needing weaker hypothesis. The proof of the existence is very similar to the one presented in [33] and it was inspired by it, the uniqueness being original. We state the problem considering an N-dimensional state where N is the age a tree dies ($f_N = 0$), but the proof in Mitra and Wan's setting of an n-dimensional state follows the same lines.

Theorem 1.2.1. *(i) Under H4-H6 the state X^* is invariant along the optimal trajectory.*

(ii) Under H4-H7 the state X^* is the unique invariant state.

Proof. (i) To see that the primal solution is

$$X_t = \big(\underbrace{\tfrac{1}{m}, \ldots, \tfrac{1}{m}}_{m}, \underbrace{0 \ldots, 0}_{N-m}\big) \text{ for all } t$$

we consider the Lagrangian

$$(1.5) \qquad \begin{aligned} L \;=\; & \textstyle\sum_{t\in I\!N} \big\{ b^t U\big(\sum_{s=1}^{N-1} f_s(x_t^s - x_{t+1}^{s+1}) + f_N x_t^N \big) + \theta_t \big(1 - \sum_{s=1}^{N} x_t^s\big) \big\} \\ & + \textstyle\sum_{t\in I\!N} \big\{ \sum_{s=1}^{N-1} \mu_t^s (x_t^s - x_{t+1}^{s+1}) + \sum_{s=1}^{N} \lambda_s^t x_t^s \big\} \end{aligned}$$

together with the following set of ℓ^1_+-multipliers

$$\begin{cases} \lambda_t^N &= \frac{b^t}{\sigma_N}\sigma_m f_m U'(\frac{f_m}{m}) \\ \mu_t^s &= \frac{b^t}{\sigma_s}(\sigma_m f_m - \sigma_s f_s) U'(\frac{f_m}{m}) \\ \theta_t &= \frac{b^t \sigma_m}{\sigma_1} f_m U'(\frac{f_m}{m}) \end{cases}$$

Due to the definition of m, the complementarity slackness and the non-negativity of all the multipliers are rightly fulfilled. A straightforward computation shows that $\nabla L = 0$ so the proposed trajectory is a stationary point for $P(X^*)$ and the optimality follows.

(ii) To see the uniqueness, let X be a state invariant. We start by proving that along the constant trajectory $z^u = x^u - x^{u+1} > 0$ implies $\sigma_u f_u \geq \sigma_v f_v$ for all v. To this end we consider the following alternative trajectory

$$\begin{aligned} \tilde{c}_t &= c + \epsilon f_v 1\!\!1_{\{t=0(v)\}} - \epsilon f_u 1\!\!1_{\{t=0(u)\}} \\ \tilde{x}_t^s &= \begin{cases} x^s & \text{if } s > \max(u,v) \\ \begin{cases} x^s - \epsilon 1\!\!1_{\{t=j(u)\}} & \text{if } u > v \\ x^s + \epsilon 1\!\!1_{\{t=j(v)\}} & \text{if } v > u \end{cases} & \text{if } \max(u,v) \geq s > \min(u,v) \\ x^s - \epsilon 1\!\!1_{\{t=j(u)\}} + \epsilon 1\!\!1_{\{t=j(v)\}} & \text{if } s \leq \min(u,v) \end{cases} \end{aligned}$$

A somewhat involved verification shows that the proposed trajectory is feasible, hence the benefit difference should be non-positive

$$\begin{aligned} 0 \geq V_\epsilon - V &= \textstyle\sum_{t\neq\dot{u}} b^{tv}[U(c+\epsilon f_v) - U(c)] + \sum_{t\neq\dot{v}} b^{tu}[U(c-\epsilon f_u) - U(c)] \\ &\quad + \textstyle\sum_{t\neq\dot{u}\dot{v}} b^{tuv}[U(c+\epsilon f_v - \epsilon f_u) - U(c)] \end{aligned}$$

Dividing by ϵ and letting $\epsilon \to 0$

$$\begin{aligned} 0 &\geq \textstyle\sum_{t\neq\dot{u}} b^{tv} f_v U'(c) - \sum_{t\neq\dot{v}} b^{tu} f_u U'(c) + \sum_{t\neq\dot{u}\dot{v}} b^{tuv}(f_v - f_u) U'(c) \\ &= f_v U'(c) \textstyle\sum_t b^{tv} - f_u U'(c) \sum_t b^{tu} = (\sigma_v f_v - \sigma_u f_u) U'(c) \end{aligned}$$

Hence, $\sigma_v f_v \leq \sigma_u f_u$. This property and H7 yield that along the stationary optimal trajectory, only z^m is different from zero. And the invariance condition gives $X = \left(\frac{1}{m}, \ldots, \frac{1}{m}, 0, \ldots, 0\right) = X^*$. ■

Besides, they propose two examples with a strictly concave U presenting a periodic optimal trajectory and hence no convergence to the normal forest, challenging a long forestry tradition. We will see in the next section that according to Salo and Tahvonen there is a relative neighborhood of X^* such that the optimal policy is periodic, and of course no convergence is possible. Even more, it is shown that these two counterexamples are in fact particular cases of their result.

In the same article, the case U linear is also treated and completely solved:

THEOREM 4.2 [22] *Let m be a Faustmann age. If the utility function U is linear and H1, H2 are fulfilled then an optimal policy consists in clearcutting every age class older or equal to m:*

$$\begin{cases} c_0 &= \sum_{s=m}^{N} f_s x_0^s \\ c_t &= f_m x_t^m, \qquad t \geq 1 \end{cases}$$

which yields an m-periodic trajectory

This policy was defined as the *Faustmann harvesting* by Salo and Tahvonen [32] and the periodic trajectory it produces is called the *Faustmann periodic solution.* Observe that the optimal policy for the normal forest is exactly the Faustmann harvesting. Notice that if (1.2) has more that one solution, there is an optimal Faustmann periodic solution for any such m.

The undiscounted case

In [23], Mitra and Wan study the forest management problem when future utilities are undiscounted ($b = 1$). Some of the main results in this framework are:

1 If U is linear, then the Faustmann harvesting periodic solution is always optimal. In this case the behavior is identical to the discounted case.

2 If U is strictly concave and increasing, any optimal solution converges to the normal forest, defined as in (1.4) where m is characterized as $m = \arg\max_s \frac{f_s}{s}$. The value m is the so-called golden rule, a name coming from the optimal growth theory and is the equivalent to the Faustmann age in the discounted case. In this framework the normal forest is known as the maximum sustained yield solution. Here, the behavior is different from the discounted case.

1.2.2 Salo and Tahvonen

More recently S. Salo and O. Tahvonen treat the problem of optimal forest management using the Karush-Kuhn-Tucker theorem of mathematical programming. Among other results, they retrieve the existence of the normal forest and prove that there is not even local convergence to such state.

They consider the same model as Mitra and Wan, with a slight difference: they require the condition $0 \leq f_1 \leq \cdots \leq f_n$ but not the concavity of the $f_i's$. They treat the case of a strictly concave U.

The authors prove that there is a set K, that includes X^*, such that the optimal policy is the Faustmann harvesting, producing an m-periodic optimal trajectory. Even more, this set is a relative neighborhood of X^*, which implies that the state X^* is not even a local attractor, so it is far from being the global attractor it was historically assumed to be. On the other hand, the same property implies that X^* is locally stable: if the initial state is close enough to it, the optimal trajectory remains in a bounded neighborhood.

PROPOSITION 1 [32] *The optimal policy for every $X \in K$ is the Faustmann harvesting*

$$(1.6)\quad K = \left\{ X \in I\!R^n_+ \Big/ \begin{array}{ll} \frac{U'(f_m x^s)}{U'(f_m x^{s-m+j})} \leq \eta_j \equiv \frac{b^{m-j} f_m}{f_j + b^m (f_m - f_j)}, & j = 1, \ldots, n,\ s = 1, \ldots, m \\ x^s = 0 & s = m+1, \cdots, n \end{array} \right\}$$

This proposition implies the existence of a invariant state. Indeed, by (1.2) it follows easily that $\eta_j \geq 1 \quad \forall j$. Hence X^* belong to K and is invariant under the Faustmann harvesting. The uniqueness of the invariant state is not treated.

Moreover, if m is unique due to the strict concavity of the function $U(\cdot)$ there is $\Phi > 0$ such that $K_\Phi \subseteq K$, where

$$K_\Phi = \left\{ \left(\tfrac{1}{m} + \Phi_1, \ldots, \tfrac{1}{m} + \Phi_m, 0, \ldots, 0 \right) \Big/ |\Phi_s| \leq \Phi, \quad \textstyle\sum_{s=1}^m \Phi_s = 0 \right\}$$

which shows that $K_\Phi \subseteq K$ is a relative neighborhood of X^*.

The proof is based on the Karush-Kuhn-Tucker theorem, where the conditions in the definition of K assure the non-negativity of the involved multipliers.

In the particular case of $n = m = 2$, Salo and Tahvonen show that every optimal trajectory reaches the set K in finite time, after which the optimal trajectory is the periodic cycle defined by the Faustmann harvesting [31]. But from the proof it is not at all evident how we could make a generalization to $n \geq m \geq 2$.

For arbitrary values of n and m the authors find a sufficient condition that assures the convergence to the set K. Consider the following finite dimension problem

$$(1.7)\qquad v(X, T) = \max \left\{ \textstyle\sum_{t=0}^{T-1} b^t U(c_t) + b^T v_K(X_T) \right\}$$

with the usual constraints for all $t = 0, \ldots, T-1$ and where $v_K(X_T) = \sigma_m \sum_{s=1}^m b^{-s} U(f_m x_T^s)$.

PROPOSITION 3 [32] *Let $X_t^*, t = 1, \ldots, T$ be the optimal sequence from X_0 for problem (1.7) subject to the usual constraints of area balance and non-negativity of state variables and* *harvests. If $X^*_{T-k} \in K$, where $k = \max(1, n-m)$, then the optimal solution to the harvesting problem with infinite horizon (1.3) is $X^*_t, t = 1, \dots, T$ continued by the Faustmann periodic solution from X^*_T.*

The proposition is indeed very useful for numerical algorithms, however, no condition assuring that such a T exists is provided.

1.2.3 Rapaport, Sraidi and Terreaux

In their article of 2003 [30], Rapaport, Sraidi and Terreaux deal with the optimal exploitation of a forest with a regenerating delay d ($d > 1$). In this model a tree is considered suitable for harvesting, or "mature" when it reaches the age d and the harvesting of younger or "immature" trees is forbidden. Furthermore, the timber content variations after maturity are neglected. This allows to represent all mature trees with only one state variable: the total area occupied with trees beyond maturity. Thus, the authors use a vector state of dimension $n = d + 1$. It also allows to reduce the complexity of the dynamics as we have now a single control variable at each time step: the land to be harvested.

The problem is studied in discrete time and all the results are presented with a continuous-space and a discrete-space version. We focus in the continuous variant.

For each period $t \in I\!N$, the area of trees of age s at time t is denoted by $x^s_t \geq 0$ and $x^{d+1}_t \geq 0$ represents the area occupied by over-mature trees (older than d). We must decide how much land $c_t \geq 0$ to harvest. Assuming that only mature trees can be harvested we must have $c_t \leq x^{d+1}_t + x^d_t$, and then the area not harvested in that period will comprise the over-mature trees at the next step, namely $x^{d+1}_{t+1} = x^{d+1}_t + x^d_t - c_t$. The evolution of the rest of the state components is an age-shift dynamics, $x^{s+1}_{t+1} = x^s_t,\ s = 1, \dots, d-1$. The harvested area is immediately allocated to new seedlings $x^1_{t+1} = c_t$.

The total benefit obtained from the harvests is given by the value $V = \sum_{t=0}^{\infty} b^t U(c_t)$, where $b \in (0, 1)$ is the discount rate and $U : I\!R \to I\!R$ is smooth, increasing and strictly concave, as before. The problem is to find the sequence of harvests $c_t \geq 0$ which maximizes the value V while keeping the state variables x^s_t non-negative subject to the above constraints.

With this model the optimization problem may be written as

$$
(\mathrm{P}_{X_0}) \left\{ \begin{array}{lll}
\underset{\boldsymbol{x}^s, \boldsymbol{c}}{\text{maximize}} & \sum_{t \in I\!N} b^t U(c_t) & \\
\text{s.a.} & x^{d+1}_{t+1} = x^{d+1}_t + x^d_t - c_t & \\
 & x^{s+1}_{t+1} = x^s_t & s = 1, \dots, d-1 \\
 & x^1_{t+1} = c_t & \\
 & x^s_0 = X^s_0 &
\end{array} \right.
$$

The existence of solution is justified by exploiting its equivalence with the solution of the Bellman's equation (see for example, Theorem 4.6 of Stokey and Lucas [36]).

Remark 1.2.2. The initial condition $(x_0^0, x_0^1, \ldots, x_0^d, x_0^{d+1})$ is completely equivalent to the initial condition $(x_0^0, x_0^1, \ldots, x_0^d + x_0^{d+1}, 0)$ as both have exactly the same feasible trajectories with identical associated benefit. Hence, without loss of generality we assume that $x_0^{d+1} = 0$.

One of the first objections to this model is that it allows trees to grow infinitely old while keeping the timber content constant, which is obviously not real. But in fact, we will see right-away that if an optimal policy is applied, trees are never allowed to grow older than $2d$. Our remark can be recovered as a generalization of a result presented in [30], where it is shown that there exists $T \in [d, 2d-1]$ such that $x_T^{d+1} = 0$, which implies that at $T-1$ the totality of the mature trees are harvested.

We claim that in every interval of length d, such as $p+1, \ldots, p+d$ there is at least one $x_t^{d+1} = 0$. Indeed, it this was not the case then at time p we could harvest a small additional area $\epsilon > 0$ modifying the trajectory as:

$$\begin{array}{ll}
\tilde{c}_p = c_p + \epsilon, & \tilde{X}_p = X_p \\
\tilde{c}_{p+1} = c_{p+1}, & \tilde{X}_{p+1} = (x_{p+1}^1 + \epsilon, x_{p+1}^2, \ldots, x_{p+1}^d, x_{p+1}^{d+1} - \epsilon) \\
\tilde{c}_{p+2} = c_{p+2}, & \tilde{X}_{p+2} = (x_{p+2}^1, x_{p+2}^2 + \epsilon, \ldots, x_{p+2}^d, x_{p+2}^{d+1} - \epsilon) \\
\quad\vdots & \quad\vdots \\
\tilde{c}_{p+d} = c_{p+d}, & \tilde{X}_{p+d} = (x_{p+d}^1, x_{p+d}^2, \ldots, x_{p+d}^d + \epsilon, x_{p+d}^{d+1} - \epsilon) \\
\tilde{c}_{p+d+1} = c_{p+d+1}, & \tilde{X}_{p+d+1} = X_{p+d+1}
\end{array}$$

after which we rejoin the original optimal trajectory from $t = p+d+1$. The new trajectory provides a strictly greater benefit, contradicting optimality of the original one. This implies that ages greater than $2d$ will never be reached along the optimal trajectory.

As in the formulation of Mitra and Wan there is a state invariant, that is called the sustainable state by the autors: $X^* = (\frac{1}{d}, \ldots, \frac{1}{d}, 0)$.

Among the set of feasible harvesting policies, the authors consider in particular the *greedy policy* defined as harvesting all the available trees at each step of time, i.e.: $c_t = x_t^{d+1} + x_t^d$. It is easily seen that the greedy policy yields a periodic trajectory of period d.

The main results obtained in this paper are:

PROPOSITION 1 [30] *The greedy policy is optimal for any initial condition iff*

$$U'(1) - bU'(0) \geq 0 \tag{1.8}$$

The following result shows that the problem can be reduced to a finite dimensional one:

PROPOSITION 2 [30] *The optimal trajectory reaches a greedy cycle in at most* $2d$ *steps.*

PROPOSITION 5 [30] *If there exists* $\hat{x} \in [\frac{1}{n}, 1]$ *such that* $U'(\hat{x}) - bU'(0) \geq 0$ *then the greedy policy is optimal from t=0 for any state* $X \in \mathcal{F}(\hat{x})$ *defined as*

$$(1.9) \quad \mathcal{F}(\hat{x}) = \left\{ X \in I\!R_+^{d+1} / \sum_{i=0}^{d+1} x^i = 1,\ x^{d+1} + x^d \leq \hat{x} \text{ and } x^s \leq \hat{x} \text{ for all } s = 1, \ldots, d-1 \right\}$$

Remark 1.2.3. If (1.8) holds, then we can take $\hat{x} = 1$, which rightly implies that the greedy policy is optimal for every possible initial state because $\mathcal{F}(1) = \{X \in I\!R_+^{d+1} / \sum_{i=0}^{d+1} x^i = 1\}$. Hence, Proposition 1 is a particular case of Proposition 5.

1.2.4 Comparing the Mitra and Wan model and the Rapaport et al. model

We claim that the Rapaport, Sraidi and Terreaux model may be regarded as a particular case of the Mitra and Wan model, and some of the results obtained by Rapaport et al. can be retrieved from those of Salo and Tahvonen. This is the case of Propositions 1 and 5 in the previous subsection. To the best of our knowledge, Proposition 2 is independent.

In order to regard the Rapaport et al. model as a model of Mitra and Wan it is enough to take $n = 2d$ and the following values for the biomass coefficients:

$$f_s = \begin{cases} 0, & s = 1, \ldots, d-1 \\ 1 & s = d, \ldots, 2d \end{cases}$$

We already know that along the optimal trajectory, no tree is allowed to grow beyond $2d$ in the Rapaport et al. formulation, thus there is no loss of generality in taking $n = 2d$ forcing the oldest age class to be clearcut.

We make the following identification between the state vectors of the two models:

$$X_{RST} = (x^1, x^2, \ldots, x^d, x^{d+1}) \longrightarrow X_{MW} = (x^1, x^2, \ldots, x^d, x^{d+1}, 0, \ldots, 0)$$
$$X_{MW} = (x^1, x^2, \ldots, x^d, x^{d+1}, \ldots, x^{2d}) \longrightarrow X_{RST} = (x^1, x^2, \ldots, x^d, \sum_{i=d+1}^{2d} x^i)$$

With these values of f_s, it is immediate that the unique Faustmann age is d and that the Mitra and Wan's normal forest and Rapaport et al's sustainable state are

$$X^*_{MW} = \Big(\underbrace{\tfrac{1}{d}, \ldots, \tfrac{1}{d}}_{d}, \underbrace{0, \ldots, 0}_{d}\Big) \qquad X^*_{RST} = \Big(\underbrace{\tfrac{1}{d}, \ldots, \tfrac{1}{d}}_{d}, 0\Big)$$

Even more, the Faustmann harvesting $(F.h.)$ is the equivalent to the greedy policy $(G.p.)$ in the sense that they both harvest the same volume of timber at each stage and their respective trajectories can be identified.

$$\begin{array}{rcll} (x^0, x^1, \dots, x^d, 0) & \longrightarrow_{G.p.} & (x^d, x^0, \dots, x^{d-1}, 0) & c = x^d \\ (x^0, x^1, \dots, x^d, 0, \dots, 0) & \longrightarrow_{F.h.} & (x^d, x^0, \dots, x^{d-1}, 0, \dots, 0) & c = x^d \end{array}$$

The following proposition, which we believe to be original, shows that Proposition 5 presented in the previous subsection [30, Proposition 5], can be retrieved from the Salo and Tahvonen, in particular from [32, Proposition 1].

Proposition 1.2.4. *$\mathcal{F}(\hat{x}) \subseteq K$, where $\mathcal{F}(\hat{x})$ is the set of initial states whose optimal policy is the greedy policy, defined in* (1.9)*, and K is the set of initial conditions whose optimal policy is the Faustmann harvesting, defined in (1.6).*

Proof. According to the particular values of f_s, the set K can be expressed as

$$K = \left\{ X_{MW} \in I\!R^{2d}_{+} \Bigg/ \begin{array}{ll} \frac{U'(x^s)}{U'(x^{s-d+j})} \leq \frac{1}{b^j} & j = 1, \dots, d,\ s = 1, \dots, d \\ \frac{U'(x^s)}{U'(x^{s-d+j})} \leq \frac{1}{b^{j-d}} & j = d+1, \dots, 2d,\ s = 1, \dots, d \\ x^s = 0 & s = d+1, \cdots, 2d \end{array} \right\}$$

Thanks to Remark 1.2.2, $\mathcal{F}(\hat{x})$ can be expressed as

$$\mathcal{F}(\hat{x}) = \{X_{RST} \in \Delta^{d+1} / \, x^{d+1} = 0 \quad \text{and} \quad x^s \leq \hat{x} \text{ for all } s = 1, \dots, d\}$$

where $\hat{x}$ is such that $U'(\hat{x}) \geq bU'(0)$. Combining the characterization of $\hat{x}$ with the concavity of U we have

$$X_{RST} \in \mathcal{F}(\hat{x}) \Rightarrow U'(x^s) \geq U'(\hat{x}) \geq bU'(0) \geq bU'(x^{s'}) \qquad \forall s, s' = 1, \dots, d$$

In particular

$$X_{RST} \in \mathcal{F}(\hat{x}) \Rightarrow \left\{ \begin{array}{ll} \frac{U'(x^s)}{U'(x^{s-d+j})} \leq b < \frac{1}{b^j} & j, s = 1, \dots, d \\ \frac{U'(x^s)}{U'(x^{s-d+j})} \leq b < \frac{1}{b} & j = d{+}1,\ s = 1, \dots, d \\ x^{d+1} = 0 & \end{array} \right. \qquad \Rightarrow X_{MW} \in K$$

■

From this result it is immediate that [30, Proposition 5] is a particular case of [32, Proposition 1] and we get also [30, Proposition] 1 thanks to Remark 1.2.3.

1.2.5 The oldest age class is harvested first

Before proceeding to many species models, we include a result we obtained with the one species Mitra and Wan model that might be part of more important developments in the future, even though it is not used in the sequel.

The following proposition states the very intuitive property that oldest age-classes should be cut first. More importantly, it reduces the problem to only one harvesting decision at each time step.

Lemma 1.2.5. *If f_s is non-decreasing and concave and $U'(f_n) > bU'(0)$, then*

$$c_t^u > 0 \Rightarrow c_t^v = x_t^u \quad \textit{for all } v > u.$$

Proof. Suppose by contradiction that there is $c_t^v < x_t^v$ with $v > u$. This would imply $x_{t+1}^{v+1} > 0$ and the following alternative trajectory would be feasible

$$\begin{array}{lll} \tilde{c}_t^u = c_t^u - \epsilon, & \tilde{c}_t^v = c_t^v + \epsilon, & \tilde{X}_{t+1} = (\dots, x_{t+1}^{u+1} + \epsilon, \dots, x_{t+1}^{v+1} - \epsilon, \dots) \\ \tilde{c}_{t+1}^u = c_{t+1}^u + \epsilon, & \tilde{c}_{t+1}^v = c_{t+1}^v - \epsilon, & \tilde{X}_{t+2} = X_{t+2} \end{array}$$

and the original trajectory follows. The difference of benefit is then

$$V_\epsilon - V = b^t[U(c_t - f_u\epsilon + f_v\epsilon) - U(c_t)] + b^{t+1}[U(c_{t+1} + f_u\epsilon - f_v\epsilon) - U(c_{t+1})]$$

Dividing by $b^t\epsilon$ and letting $\epsilon \to 0$ we get

$$U'(c_t)(f_v - f_u) - bU'(c_{t+1})(f_{u+1} - f_{v+1}) \geq [U'(c_t) - bU'(c_{t+1})](f_v - f_u) > 0$$

contradicting optimality. ∎

1.3 Multiple Species

We present now a study by Salo and Tahvonen [31, 33] of a single species forest combined with an annual alternative use of the land, a case that may be considered as a particular two species forest. Using a generalization of the model of Mitra and Wan, they prove the existence of an invariant state and some periodic optimal trajectories. They also tackle the problem of the stability of the invariant state, proving that in some cases it is a local saddle point. Finally they present a numerical example where the optimal trajectory converges to it.

Remark 1.3.1. In the following we will call *sustainable* to any state invariant under an optimal policy. With this definition Mitra and Wan's normal forest and Rapaport et al's sustainable state are *sustainable states* of their corresponding mono-species models.

From the following chapter on, we will consider the generalization to multiple species of the Rapaport et al. model with no restriction on the feasible trajectories. In fact, we will study a model halfway between the general multi-species Mitra and Wan model and its restriction to Faustmann harvesting policies. We prove that for a multiple-species forest there is always one unique sustainable state. We also prove that there is convergence to the sustainable state of any optimal trajectory, as long as the maturity ages of the species present at the sustainable state are coprimes. Of course, the condition is fulfilled if there is an annual alternative use of the land.

However, the generalization of the Mitra and Wan model considering every trajectory is very difficult to treat, we present at the end of this section the proof of the existence and uniqueness of the sustainable state in this framework. But, we think it may be very difficult to go beyond that point.

1.3.1 Two species: Salo and Tahvonen

In Salo and Tahvonen articles of 2002 and 2004 [31, 33], an alternative annual use of the land is taken into consideration (for example, agriculture use or allocation to a third party). We call y_t the fraction of the land allocated to this alternative use at time t and $W(\cdot)$ the benefit function associated. The state is now represented by $\mathbb{X}_t = \big((x_t^1, \ldots, x_t^n), y_t\big)$. We still denote by m the Faustmann age of the forest species X and it is assumed to be unique.

Assuming that the two activities provide independent benefit and that there are no costs associated to the land conversion between them, the new problem is

$$
(\mathrm{P}_{\mathbb{X}_0}) \begin{cases} \max & \sum_{t \in I\!N} b^t \left[U(c_t) + W(y_t) \right] \\ \text{s.a.} & c_t = \sum_{s=1}^{n-1} f_s(x_t^s - x_{t+1}^{s+1}) + f_n x_t^n \\ & y_t = 1 - \sum_{s=1}^{n} x_t^s \\ & x_t^s - x_{t+1}^{s+1} \geq 0 \qquad\qquad s = 1, \ldots, n-1 \\ & x_t^s \geq 0 \qquad\qquad \forall s \\ & y_t \geq 0 \\ & x_0^s = X_o^s \qquad\qquad \forall s \end{cases}
$$

where the benefit functions $U(\cdot)$ and $W(\cdot)$ are assumed smooth, increasing and strictly concave and $b \in (0,1)$, as usual. We can eliminate the variables c_t and y_t to state the problem as

$$
(\mathrm{P}_{\mathbb{X}_0}) \begin{cases} \underset{\boldsymbol{x}^s}{\text{maximize}} & \sum_{t \in I\!N} b^t \left[U\left(\sum_{s=1}^{n-1} f_s(x_t^s - x_{t+1}^{s+1}) + f_n x_t^n\right) + W\left(1 - \sum_{s=1}^{n} x_t^s\right) \right] \\ \text{s.a.} & x_t^s - x_{t+1}^{s+1} \geq 0 \qquad s = 1, \ldots, n-1 \\ & 0 \leq 1 - \sum_{s=1}^{n} x_t^s \\ & x_0^s = X_0^s \qquad \forall s \end{cases}
$$

The existence of optimal solutions for every initial condition, follows again from Stokey and Lucas [36, Theorem 4.6]. Associated to problem $(P_{\mathbb{X}_0})$ we have the following Lagrangian

$$(1.10)\quad L(\boldsymbol{x}^s,\lambda,p)=\textstyle\sum_{t\in I\!N}\big\{b^t[U(c_t)+W(y_t)]+\lambda_t\big(1-\sum_{s=1}^{N}x_t^s\big)+\sum_{s=1}^{n-1}p_t^s(x_t^s-x_{t+1}^{s+1})\big\}$$

where we keep the notation $c_t=\sum_{s=1}^{n-1}f_s(x_t^s-x_{t+1}^{s+1})+f_nx_t^n$ and $y_t=1-\sum_{s=1}^{n}x_t^s$. The Karush-Kuhn-Tucker necessary conditions for all $t\in I\!N$ are

$$(1.11)\quad b^{-t}\frac{\partial L}{\partial x_{t+1}^1}=bf_1U'(c_{t+1})-bW'(y_{t+1})-\lambda_t+bp_{t+1}^1\le 0$$
$$(1.12)\quad b^{-t}\frac{\partial L}{\partial x_{t+1}^{s+1}}=bf_{s+1}U'(c_{t+1})-f_sU'(c_t)-bW'(y_{t+1})-\lambda_t+bp_{t+1}^{s+1}-p_t^s\le 0,\quad 1\le s\le n-2$$
$$(1.13)\quad b^{-t}\frac{\partial L}{\partial x_{t+1}^n}=bf_nU'(c_{t+1})-f_{n-1}U'(c_t)-bW'(y_{t+1})-\lambda_t-p_t^{n-1}\le 0$$
$$(1.14)\quad x_{t+1}^s\ge 0,\qquad x_{t+1}^s\frac{\partial L}{\partial x_{t+1}^{s+1}}=0\qquad 1\le s\le n$$
$$(1.15)\quad p_t^s\ge 0,\qquad p_t^s(x_t^s-x_{t+1}^{s+1})=0\qquad 1\le s\le n-1$$
$$(1.16)\quad \lambda_t\ge 0,\qquad \lambda_t\big(1-\textstyle\sum_{s=1}^{n}x_{t+1}^s\big)=0$$

On the existence of stationary cycles

The authors first study the existence of states whose optimal trajectory is the Faustmann periodic solution with no land ever allocated to the alternative use.

PROPOSITION 1 [33] *Given $g\equiv f_m\sigma_mU'(\frac{f_m}{m})-\sigma_1W'(0)>0$ and $m\ge 2$ there exists $0<\Phi<\frac{1}{m}$ such that the optimal trajectory for every $\mathbb{X}\in K_\Phi$ is the Faustmann periodic solution with $y_t\equiv 0$.*

$$K_\Phi=\big\{\mathbb{X}\in I\!R_+^{n+1}/\,\mathbb{X}=\big((\tfrac{1}{m}+\Phi_1,\dots,\tfrac{1}{m}+\Phi_m,0,\dots,0),0\big)\text{ with }|\Phi_s|<\Phi\text{ and }\textstyle\sum_s\Phi_i=0\big\}$$

To show that the Faustmann periodic trajectory is an optimal solution of the concave problem $(P_{\mathbb{X}_0})$ with $\mathbb{X}_0\in K_\Phi$, it suffices to show that it is a stationary point of the Lagrangian (1.10). To this end, the authors give values to the multipliers p_t^s and λ_t satisfying (1.11)-(1.16) and the proposition follows because the transversality condition $\lim_{t\to\infty}b^t\partial U/\partial x_t^s\cdot x_t^s=0, s=1,\dots,n$ is trivially fulfilled by any bounded solution $\boldsymbol{x}^s$. For details we refer the reader to [33].

We make the observation that if $g>0$ the state $\mathbb{X}^*=\big((\frac{1}{m},\dots,\frac{1}{m},0,\dots,0),0\big)$ is invariant. This is evident because $\mathbb{X}^*\in K_\Phi$ and the Faustmann harvesting yields a constant trajectory.

COROLLARY 1 [33] *If $g\le 0$, the only optimal periodic trajectory with $y_t=y^*>0$ for all t is the constant trajectory, $x_t^s=\frac{1-y^*}{m}$ for $s\le m$ and $x_t^s=0$ for $s>m$.*

Next, it is established the existence of a unique invariant state whenever $g \leq 0$. Such state is exactly $\mathbb{X}^* = \big((\frac{1-y^*}{m}, \ldots, \frac{1-y^*}{m}, 0, \ldots, 0), y^*\big)$ where y^* is defined as the unique solution of

$$f_m \sigma_m U'\big(f_m \tfrac{(1-y^*)}{m}\big) - \sigma_1 W'(y^*) = 0.$$

Notice that if $g = 0$, then we retrieve $\bar{\mathbb{X}}$ as the invariant state and that if $\sigma_1 W'(1) > f_m \sigma_m U'(0)$ holds then we would have $y^* > 1$ which evidently does not yield a valid state. In such situation, it is possible to prove that $\mathbb{X}^* = \big((0, \ldots, 0), 1\big)$ is the unique invariant state. Since the authors do not treat this case, in the following discussion we will assume $y^* \in (0,1)$ to rule it out.

Stability of the sustainable state

After having characterized the sustainable state, the authors consider the more difficult task of studying its stability. We already know that there is no local convergence of the optimal trajectory to this state when $g > 0$. So, the case $g \leq 0$ ($y^* > 0$) is next considered. The authors prove that the sustainable state is a local saddle point for every rate of discount.

The resolution is somewhat involved. Using the fact that along a Faustmann harvesting policy $x_t^s = x_{t+1}^{s+1}, \quad s = 1, \ldots, m-1$ and $x_t^s = 0, \quad s = m+1, \ldots, n$ a single $(2m-2)$-order difference equation for x_t^m can be deduced from equations (1.11)-(1.13), namely

(1.17) $$b^{m-1} f_m U'(f_m x_{t+m-1}^m) - \sum_{i=0}^{m-1} b^i W'(y_{t+i}) = 0, \quad \text{where } y_t = 1 - \sum_{j=0}^{m-1} x_{t+j}^m.$$

The first order approximation of (1.17) in a neighborhood of the constant trajectory ($u_t = x_t^m - x^* \approx 0$) yields

$$b^{m-1} f_m^2 \frac{U''(f_m x^*)}{W''(y^*)} u_{t+m-1} + \sum_{i=0}^{m-1} b^i W''(y^*) \sum_{j=0}^{m-1} u_{t+i+j} = 0$$

whose corresponding characteristic polynomial Ω is

$$\Omega(u) = u^{2m-2} + \sum_{i=1}^{m-2} \frac{b^{-i} - b}{1-b} u^{2m-2-i} + \Big(f_m^2 \frac{U''(f_m x^*)}{W''(y^*)} + \frac{b^{1-m} - b}{1-b}\Big) u^{m-1} + \sum_{i=m}^{2m-2} \frac{b^{1-m} - b^{m-i}}{1-b} u^{2m-2-i}$$

Salo and Tahvonen prove:

LEMMA 1 [33] *The roots of Ω consist of pairs α_i, α_{i+m-1} with $\alpha_{i+m-1} = \frac{1}{b\alpha_i}, i = 1, \ldots, m-1$. In addition, if $f_m^2 \frac{U''(f_m x^*)}{W''(y^*)} \geq 0$, $b \leq 1$ and $m \geq 1$ then $\Omega(1) > 0$.*

LEMMA 2 [33] *If* $f_m^2 U''/W'' > 0$*,* $b \leq 1$ *and* $m \geq 1$ *then* $m-1$ *of the roots have moduli less that 1 and* $m-1$ *have moduli greater than 1.*

Lemma 1 and 2 readily imply that the sustainable state is a local saddle point. Some numerical simulations where there is convergence towards $\mathbb{X}^*$ are presented with free end point conditions and finite time horizon. According to the authors, the finite horizon is taken long enough to approximate the infinite horizon solution. These simulations suggest that the optimal trajectory converges to $\mathbb{X}^*$ from *any* initial state.

And finally the authors extend the model to consider conversion costs between forestry and the alternative use of the land. They assumed these costs to be linear with respect to the transferred areas. Two numerical examples are presented in which the optimal trajectory is a periodic cycle around the sustainable state, even if $y^* > 0$, a situation forbidden in the simpler case with no conversion costs.

1.3.2 Extension of the Mitra and Wan model for multiple species: sustainable state

In this subsection we prove the existence of the sustainable state considering the Mitra and Wan model adapted to a multi-species forest. This represents a generalization of the existence of sustainable state in the case of a unique species forest with alternative use due to Salo and Tahvonen that we reviewed in the previous subsection. We include the proof of the uniqueness of such a state. However, we do not study the existence of optimal periodic cycles.

We consider a forest of total area S occupied by k-species $I = \{1, \ldots, k\}$, with biomass functions $f^i : \mathbb{N} \to \mathbb{R}_+$ and benefit functions $U_i : \mathbb{R}_+ \to \mathbb{R}_+$ respectively. Let m_i be a Faustmann age of each of the species, i.e., $m_i = \arg\max_s \sigma_s f_s^i$. Without loss of generality we assume that the species are ordered in such a way that $\sigma_1 f_{m_1}^1 U_1'(0) \geq \sigma_2 f_{m_2}^2 U_2'(0) \geq \cdots \geq \sigma_k f_{m_k}^k U_k'(0)$

We prove the existence of the sustainable state and its uniqueness whenever the m_i are unique for all i. We recall that in the one species problem, the sustainable state would be $(\frac{S}{m_i}, \ldots, \frac{S}{m_i}, 0, \ldots, 0)$ for each one of the species. In this framework, such a state will be the natural extension of this configuration: every species will be of the form $X^i = (\underbrace{x^i, \ldots, x^i}_{m_i}, 0, \ldots, 0)$.

The benefit obtained from the harvest is $\sum_{i\in I}\sum_{t=0}^{\infty} b^t U_i(c_t^i)$ where U_i are smooth, increasing and strictly concave. Given the initial state $X_0^i = (x_0^{1i}, \ldots, x_0^{N_i i})$, and denoting $\boldsymbol{x}^{si}$ and $\boldsymbol{z}^{si}$ the sequences of states and controls all of which belong to $\ell^\infty(\mathbb{N})$ the optimization problem may be stated as

$$
(\mathrm{P}_{X_0}) \begin{cases} \text{Max} & \sum_{t\in I\!N} b^t \left[\sum_{i\in I} U_i(c_t^i)\right] \\ \text{s.t.} & c_t^i = \sum_{s=1}^{N_i} f_s^i z_t^{si} \\ & x_{t+1}^{s+1,i} = x_t^{si} - z_t^{si} \qquad\qquad s = 1,\ldots,N_i - 1 \\ & z_t^{N_i i} = x_t^{N_i i} \\ & \sum_{i\in I}\sum_{s=1}^{N_i} x_t^{si} = 1 \\ & x_0^s = X_0^s \\ & \boldsymbol{x}^{si}, \boldsymbol{z}^{si}, \boldsymbol{c}^s \in \ell_+^\infty \end{cases}
$$

Eliminating variables c_t^i and z_t^{si} the problem can be written as

$$
(\mathrm{P}_{X_o}) \begin{cases} \underset{\boldsymbol{x}^{si}\in\ell_+^\infty}{\text{maximize}} & \sum_{t\in I\!N} b^t \sum_{i\in I} U_i\big(\sum_{s=1}^{N_i-1} f_s^i (x_t^{si} - x_{t+1}^{s+1,i}) + f_{N_i} x_t^{N_i}\big) \\ \text{s.t.} & \sum_{i=1}^{k}\sum_{s=1}^{N_i} x_t^{si} = 1 \\ & x_t^{si} - x_{t+1}^{s+1,i} \geq 0 \qquad \forall s = 1,\ldots,N_i - 1 \\ & x_0^{si} = X_0^{si} \end{cases}
$$

Consider the strictly convex program

$$
(S) \quad \begin{cases} \max & \sum_{i\in I} n_i \sigma_i f_{m_i}^i U_i(x^i) \\ \text{s.t. } x^i \geq 0 \text{ and } \sum_{i\in I} n_i x^i = S. \end{cases}
$$

Let $\mathbb{X}^*$ be a state of the form $X^i = (\underbrace{x^i,\ldots,x^i}_{m_i},0,\ldots,0)$ where $x^i = x^{*i}$ is the optimal solution to (S).

Denote $I^* = \{i\in I : x^{*i} > 0\}$ the species which are present in $\mathbb{X}^*$ and let r be the Lagrange multiplier associated to the area constraint $\sum_{i\in I} n_i x^i = S$, so that the optimal solution is characterized by $\sigma_i f_{m_i}^i U_i'(x^{*i}) = r$ for $i \in I^*$ and $\sigma_j f_{m_j}^j U_j'(0) \leq r$ for $j \notin I^*$. The ordering of the species and the strict concavity of U_i then imply that whenever $x^{*i} > 0$ we must also have $x^{*j} > 0$ for all $j < i$, so that $I^* = \{1,\ldots,i^*\}$ for some index i^*.

Proposition 1.3.2. *The state $\mathbb{X}^*$ defined above is sustainable. Furthermore, it is the unique sustainable state whenever the m_is are unique.*

Proof. Let us first show that the stationary trajectory $x_t^{si} = x^{*i}$, if $s \leq m_i$ and $x_t^{si} = 0$ otherwise, is optimal for $P(\mathbb{X}^*)$. To this end we use the Karush-Kuhn-Tucker theorem of mathematical programming considering the following Lagrangian

$$
\begin{aligned} L \;=\; & \sum_{t\geq 0}\sum_{i\in I} b^t U\big(\sum_{s=1}^{N_i-1} f_s^i (x_t^{si} - x_{t+1}^{s+1,i}) + f_{N_i} x_t^{N_i}\big) \\ & + \sum_{t\geq 1}\big[\sum_{i\in I}\sum_{s=1}^{N} \lambda_t^{si} x_t^{si} + \theta_t\big(1 - \sum_{i\in I}\sum_{s=1}^{N_i} x_t^{si}\big)\big] \\ & + \sum_{t\geq 0}\sum_{i\in I}\sum_{s=1}^{N_i-1} \mu_t^{si}(x_t^{si} - x_{t+1}^{s+1,i}) \end{aligned} \tag{1.18}
$$

together with the following set of ℓ^1-multipliers

$$\begin{cases} \theta_t &= \frac{b^t r}{\sigma_1} \\ \lambda_t^{si} &= 0 \qquad s < N_i \\ \lambda_t^{N_i i} &= \frac{b^t}{\sigma_{N_i}} \sigma_{m_i} f_{m_i} U_i'(f_{m_i}^i x^i) \\ \mu_t^{si} &= \frac{b^t}{\sigma_{s_i}} \left[\sigma_{m_i} f_{m_i}^i - \sigma_s f_s^i\right] U_i'(f_{m_i}^i x^i) \end{cases}$$

Due to the definition of m_i the complementarity slackness and the non-negativity of all the multipliers are rightly fulfilled.

It is only left to see the stationarity of the Lagrangian (1.18). The partial derivatives with respect to x_t^{si} are for all $t \geq 1$

$$\begin{aligned} L_{x_t^{i1}} &= b^t f_1^i U_i'(f_{m_i}^i x^i) + \lambda_t^{i1} + \mu_t^{i1} - \theta_t \\ L_{x_t^{is}} &= (b^t f_s^i - b^{t-1} f_{s-1}^i) U_i'(f_{m_i}^i x^i) + \lambda_t^{si} + \mu_t^{si} - \mu_{t-1}^{s-1,i} - \theta_t \qquad s = 2, \ldots, N_i - 1 \\ L_{x_t^{N_i}} &= -b^{t-1} f_{N-1}^i U_i'(f_{m_i}^i x^i) + \lambda_t^{N_i,i} - \mu_{t-1}^{N_i-1,i} - \theta_t \end{aligned}$$

A straightforward computation shows that all the partial derivatives are zero, and the optimality of the stationary trajectory follows.

To prove the uniqueness, let us suppose that there exists a state $\mathbb{X}$ sustainable and different from $\mathbb{X}^*$. The results for a unique species forest forces $\mathbb{X}$ to be of the form

$$X^i = \big(\underbrace{\tfrac{S_i}{m_i}, \ldots, \tfrac{S_i}{m_i}}_{m_i}, \underbrace{0, \ldots, 0}_{N_i - m_i}\big) \quad \text{where} \quad \textstyle\sum_{i \in I} S_i = 1.$$

It only rests to see that the vector $x = (\frac{S_1}{m_1}, \ldots, \frac{S_k}{m_k})$ is the optimal solution to (S) to conclude that $\mathbb{X} = \mathbb{X}^*$.

We claim that when $S_i > 0$ then $\sigma_i f_{m_i}^i U_i'(x^i) \geq \sigma_j f_{m_j}^j U_j'(x^j)$ for all $j \in I$. Indeed, let us perturb the optimal harvesting policy as follows: at time $t = 0$ we sow $x^i - \epsilon$ and $x^j + \epsilon$ instead of x^i and x^j, while in all subsequent periods we continue with a Faustmann harvesting policy, sowing the harvested areas with the same species they had. The benefit V_ϵ derived from this perturbed policy must be less than the value V obtained with the optimal one, which gives

$$V_\epsilon - V = \tfrac{b^{m_i}}{1-b^{m_i}} [U_i\big(f_{m_i}^i (x^i - \epsilon)\big) - U_i(f_{m_i}^i x^i)] + \tfrac{b^{m_j}}{1-b^{m_j}} [U_j\big(f_{m_j}^j (x^j + \epsilon)\big) - U_j(f_{m_j}^j x^j)] \leq 0.$$

Dividing by ϵ and letting $\epsilon \to 0$ we deduce $\sigma_i f_{m_i}^i U_i'(x^i) \geq \sigma_j f_{m_j}^j U_j'(x^j)$ as claimed. Setting $I^* = \{i : x^i > 0\}$ it follows that $\sigma_i U'(x^i)$ is constant for $i \in I^*$ and larger than the value of this expression for $i \notin I^*$. This implies that the vector $(x^i)_{i \in I}$ is an optimal solution to (S) so that $x^i = x^{*i}$ completing the proof. ∎

1.4 Summary and perspectives

In this review we have considered only works of forest economics where *no* assumption was made on the long run behavior of the forest. We have seen that the existing studies comprise solving variations of a dynamic optimization problem with age classes structure. Such variations are retrieved by changing the growth function and the constraints imposed to harvesting decisions.

Most of the results consider unique-species forests, with the exception of [31, 33] where a particular two-species forest (where one of the species is annual) is studied.

The main results can be summarized as:

- Existence of the sustainable state and its uniqueness under some conditions. This state is not a global attractor for a unique-species forest and some counterexamples are presented. In the case of a forest with an annual alternative use, this state is proved to be a local saddle point whenever it allocates part of the land to the alternative use. Some numerical examples where the optimal trajectory converges towards it are presented.
- Discussion of the conditions under which there exist greedy periodic cycles and the characterization of a subset of them, which turns out to be a neighborhood of the sustainable state.

The techniques used depend strongly on the number of species and age classes and they are not suitable to go beyond a two-species forests where one of the species is annual. However, the results obtained in Subsection 1.3.2 suggest that the theory may be extended to a forest composed by multiple species with different growth functions.

In the following three chapters we study one extension in great detail. In Chapter 2, we characterize completely the asymptotic behavior of an optimally managed multiple-species forest having different maturity ages. In Chapter 3, we take the land market into account, introducing the possibility of trading land at every step. Finally, in Chapter 4, we adapt the model to force leaving the land in fallow after the harvest, which drives the system into a completely different behavior.

Chapter 2

Asymptotic convergence of optimal harvesting policies for a multiple species forest

This chapter corresponds to the article *Asymptotic convergence of optimal harvesting policies for a multiple species forest* [1], submitted.

Abstract

We study the asymptotic behavior of the optimal harvesting policies for a mixed forest with multiple species having different maturity ages. We prove the existence and uniqueness of a *sustainable state* and we discuss the conditions under which an optimal trajectory converges in the long run towards this state or towards an optimal periodic cycle. We also analyze different situations under which the convergence occurs in finite time.

Key words. forest management, discrete dynamic programming, infinite horizon, asymptotic convergence, Lyapunov stability

AMS subject classification. 93C55, 93D05, 93D20, 90B50

2.1 Introduction

In 1849, Martin Faustmann stated the problem of finding the economic value of an even-aged forest stand. Considering an infinite horizon discrete time model and periodic policies, he obtained an expression for the present value of the stand and suggested that this formula could be used to determine an optimal harvesting age [13]. The question was solved by Ohlin in 1921 characterizing the optimal rotation period, now known as the *Faustmann age* [26]. The elegance and simplicity of the result stems from the fact that we deal with a forest of identically aged trees. The generalization of the optimal rotation problem to a forest with many even-aged stands was already considered at that time, but its complete resolution remains open even today. Nevertheless, Faustmann's ideas were extremely influential and inspired various harvesting rules with a long run behavior that guarantees a sustainable and regular flow of timber. In particular, optimal harvesting policies have been studied numerically with different types of even-flow constraints, or requiring convergence to a steady state such as the so-called *normal forest* in which the land is evenly allocated among all the age classes with a Faustmann rotation period.

More recently, the optimal harvesting problem was reconsidered by Mitra and Wan [22] as a dynamic optimization problem, with results that partially contradict the steady state paradigm. Although their main result proved the existence of a sustainable state which is invariant under an optimal policy, they also found examples where the optimal solution is a periodic cycle and *does not* converge to the sustainable state. The issue was further investigated by Salo and Tahvonen [32] showing that every initial condition close enough to the sustainable state yields a periodic optimal trajectory so that this state is not even a local attractor. They conjectured that the long run behavior of any optimal harvesting policy would be periodic, proving this result for the case of a two stand forest [31]. The same authors extended the model to include the possibility of allocating some land to an annual alternative use, showing that the optimal periodic cycles disappear when it is optimal to allocate part of the land to the annual use [31, 33]. In this setting they prove that the sustainable state is a local attractor saddle point. The optimal management of a one species forest was also studied by Rapaport, Sraidi and Terreaux [30] using a model where only mature trees older than a certain age may be harvested. They defined a *greedy policy* as one where each tree is harvested as soon as it reaches maturity, showing that every optimal trajectory becomes greedy and periodic after a finite time.

In this paper we consider the extension of the Rapaport-Sraidi-Terreaux model to the case of a mixed forest composed by several species with different maturity ages. If we restrict to one or two species and greedy policies, the model also becomes equivalent to the one by Salo and Tahvonen provided that harvesting of young vintages is forbidden. We generalize the results of Salo and Tahvonen for two competing uses of the land, by considering multiple species with arbitrary maturity ages and using a different methodology based on Lyapunov functions that allow to establish global convergence results. More precisely, we prove the asymptotic convergence towards the sustainable state whenever the maturity ages of the species present at this state are co-prime, and convergence towards the set of greedy periodic cycles otherwise. In the special case of an annual alternative use we recover the results for the two species model of Rapaport-Sraidi-Terreaux.

Asymptotic convergence of optimal trajectories, also known as *turnpike properties*, have been established for wide classes of dynamic optimization problems, most of them issued from the literature on economic growth models. For a survey of the general results available we refer to [16, 18, 19, 38] and references therein. A distinguishing feature in our context is that we obtain global convergence in a discounted utility framework with no restriction on the discount factor, while most turnpike theorems are either local or assume discount factors close to one. This is a notable fact since dynamic optimization models under strong discounting often exhibit a complex behavior including chaos [5, 16, 17, 21, 24]. A factor that explains this more regular behavior is that the forest evolution has a natural periodic structure determined by the least common multiple N of the maturity ages of the species involved. This allows to construct a Lyapunov function which does not increase at every time step but every N periods, leading nevertheless to asymptotic convergence.

The paper is structured as follows. Section §2.2 introduces the optimization model to be solved and then §2.3 describes some periodic optimal trajectories including a precise definition of the sustainable state and the greedy periodic cycles. In §2.4 we state our main results on the convergence of an optimally managed forest towards the sustainable state when this state allocates the land to species whose maturity ages are co-prime, and convergence towards a greedy periodic cycle otherwise. Finally, in §2.5 we discuss the finite time convergence for a two species forest.

2.2 Model formulation

We consider a discrete time model for the optimal management of a forest of total area S occupied by k species $I = \{1, \dots, k\}$ with maturity ages of $n_1, \dots, n_k$ years respectively. For each period $t \in \mathbb{N}$ we denote $x^i_t \geq 0$ the area of species $i \in I$ that reaches its maturity in year t, and $\bar{x}^i_t \geq 0$ the area occupied by over-mature trees (older than n_i). We must decide how much land $u^i_t \geq 0$ to harvest and how to reallocate this land to new seedlings. Assuming that only mature trees can be harvested we must have $u^i_t \leq \bar{x}^i_t + x^i_t$, and then the area not harvested in that period will comprise the over-mature trees at the next step, namely

$$\bar{x}^i_{t+1} = \bar{x}^i_t + x^i_t - u^i_t. \tag{2.1}$$

The total harvested area $\sum_{i \in I} u^i_t$ is allocated to new seedlings that will reach maturity in years $t + n_i$ respectively, which is expressed by the equation

$$\textstyle\sum_{i \in I} x^i_{t+n_i} \;=\; \sum_{i \in I} u^i_t. \tag{2.2}$$

The total benefit obtained from the harvests is given by the value

$$V = \sum_{i\in I} \sum_{t=0}^{\infty} b^t U_i(u_t^i) \tag{2.3}$$

where $b \in (0,1)$ is a discount rate and $U_i : I\!R \to I\!R$ is smooth, increasing and strictly concave for each $i \in I$. The problem is to find the sequence of harvests $u_t^i \geq 0$ which maximizes the value (2.3) while keeping the state variables x_t^i and $\bar{x}_t^i$ non-negative subject to the constraints (2.1) and (2.2).

We denote $\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i$ the sequences of states and controls. Since all the areas are non-negative and smaller than S these sequences belong to the ℓ^∞ ball B_S^∞ of radius S and centered at the origin. An alternative representation of the forest in terms of the age distribution at time t is provided by the *state* $\mathbb{X}_t = (X_t^1, \ldots, X_t^k)$ where $X_t^i = (x_{t+n_i-1}^i, x_{t+n_i-2}^i, \ldots, x_t^i, \bar{x}_t^i)$ describes the areas occupied in year t by trees of species i with ages $1, 2, \ldots, n_i$ and over n_i. The state evolution consists of an age-shift dynamics, except for the first and last components of each vector X_t^i which are controlled by the sowing and harvesting policies. Although we will not use these dynamics explicitly, the state $\mathbb{X}_t$ will be useful in describing the asymptotic behavior of the forest. Notice that we do not control $\mathbb{X}_0$ which corresponds to the initial state reflecting the age-class composition of the forest at time $t = 0$, so that the problem to be solved may be stated as

$$P(\mathbb{X}_0) \quad \begin{cases} \text{maximize} \quad V = \sum_{i\in I} \sum_{t=0}^{\infty} b^t U_i(u_t^i) \\ \text{subject to (2.1) and (2.2)} \\ \boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i \in \ell_+^\infty \text{ with } \mathbb{X}_0 \text{ given.} \end{cases}$$

We denote Δ the set of all initial states $\mathbb{X}_0$ such that $\sum_{i\in I}[\bar{x}_0^i + \sum_{t=0}^{n_i-1} x_t^i] = S$. Clearly, the area balance constraints imply that $\mathbb{X}_t \in \Delta$ for all $t \in I\!N$. We also denote Δ^0 the set of states with $\bar{x}_0^i = 0$ for all $i \in I$, and we observe that an initial state $\mathbb{X}_0 \in \Delta$ yields the same optimal value and harvesting policy as $\tilde{\mathbb{X}}_0 \in \Delta^0$ where $\tilde{X}_0^i = (x_{n_i-1}^i, \ldots, x_1^i, x_0^i + \bar{x}_0^i, 0)$.

Intuitively, the presence of a discount factor suggests an advantage for control strategies that harvest all the mature trees as soon as possible, that is to say, an optimal solution should lead to $\mathbb{X}_t \in \Delta^0$. However, this is balanced by the concavity of U_i which favors homogeneous harvests at each stage. Hence, keeping some trees beyond maturity may still be convenient in order to transfer area between different age-classes and reshape the forest into a more homogeneous state. The difficulty in solving $P(\mathbb{X}_0)$ comes precisely from the trade-offs among these two conflicting forces. A natural conjecture is that the area transfers should occur only during an initial phase after which the optimal policy should drive the forest to an homogeneous state. While this is not always the case, in the next sections we investigate the asymptotic behavior of the state $\mathbb{X}_t$ for an optimally managed forest. Before proceeding we discuss briefly the existence and uniqueness of optimal policies. The arguments are standard so the reader may wish to skip the proofs.

Proposition 2.2.1. *For each $\mathbb{X}_0 \in \Delta$ the problem $P(\mathbb{X}_0)$ has optimal solutions.*

Proof. The feasible set of $P(\mathbb{X}_0)$ is non-empty as it follows by considering the periodic trajectory that results from a control strategy in which all mature areas are harvested and then sowed with the same species as before, that is to say, $u^i_t = \bar{x}^i_t + x^i_t = x^i_{t+n_i}$. On the other hand, we already observed that $\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i \in B^\infty_S$ and since this ball is $\sigma(\ell^\infty, \ell^1)$-compact while the linear constraints in $P(\mathbb{X}_0)$ define a closed convex subset of $(B^\infty_S)^{3k}$, we deduce that the feasible set is weak* compact. Hence it suffices to prove that the objective function is weak* upper semi-continuous. The functions $u^i \mapsto \sum_{t=0}^\infty b^t U_i(u^i_t)$ are concave and strongly continuous from ℓ^∞ to $I\!R$, and therefore they are weakly u.s.c.. However we must consider the weak* topology so we proceed directly. Consider a weak* convergent net $\boldsymbol{u}^\alpha \overset{*}{\rightharpoonup} \boldsymbol{u}^i$ so that $u^\alpha_t \to u^i_t$ for all $t \in I\!N$, and take $\beta, \delta \in I\!R$ with $U_i(z) \leq \beta + \delta z$. For any fixed $N \in I\!N$ we may write

$$\begin{aligned} \limsup_\alpha \textstyle\sum_{t=0}^\infty b^t U_i(u^\alpha_t) &= \textstyle\sum_{t=0}^N b^t U_i(u^i_t) + \limsup_\alpha \sum_{t=N+1}^\infty b^t U_i(u^\alpha_t) \\ &\leq \textstyle\sum_{t=0}^N b^t U_i(u^i_t) + \limsup_\alpha \sum_{t=N+1}^\infty b^t [\beta + \delta u^\alpha_t] \\ &= \textstyle\sum_{t=0}^N b^t U_i(u^i_t) + \sum_{t=N+1}^\infty b^t [\beta + \delta u^i_t] \end{aligned}$$

the last equality since $(b^t \delta)_{t \in I\!N} \in \ell^1(I\!N)$ and $\boldsymbol{u}^\alpha \overset{*}{\rightharpoonup} \boldsymbol{u}^i$. Letting $N \to \infty$ we conclude

$$\limsup_\alpha \textstyle\sum_{t=0}^\infty b^t U_i(u^\alpha_t) \leq \sum_{t=0}^\infty b^t U_i(u^i_t).$$

■

The strict concavity of the U_i's implies that the harvests $\boldsymbol{u}^i$ are uniquely determined, though uniqueness of $\boldsymbol{x}^i$ and $\bar{\boldsymbol{x}}^i$ is not clear. However, if we restrict to greedy trajectories in which $\bar{\boldsymbol{x}}^i = 0$, then uniqueness of $\boldsymbol{x}^i$ follows as well. The latter remains true even if one utility function, say U_{i_0}, is merely concave instead of strictly concave. Indeed, using (2.2) we may eliminate the variables $u^{i_0}_t$ and write a reduced optimization problem which gives the uniqueness of the harvests $\{\boldsymbol{u}^i : i \neq i_0\}$. If we then consider greedy strategies, all the area flows $\boldsymbol{x}^i$ and $\boldsymbol{u}^i$ (including $i = i_0$) will be uniquely determined.

The existence of multipliers in ℓ^1 (*i.e.* a solution for a dual problem) is a delicate issue since we lack a constraint qualification. However, in the next sections we consider situations where the optimal strategy is periodic and for which one can explicitly find multipliers.

2.3 Stationary optimal trajectories

Some initial states $\mathbb{X}_0$ lead to optimal trajectories which are either *invariant* or *periodic*. Our goal in later sections is precisely to understand the extent to which an optimally managed forest starting from an arbitrary initial condition may or may not converge to such optimally stationary states.

2.3.1 Sustainable state

We begin by introducing the notion of a sustainable state, which corresponds intuitively to a forest with an age distribution at which it is optimal to stay forever.

Definition 2.3.1. A state $\mathbb{X} \in \Delta$ is called *sustainable* if it is invariant under an optimal policy.

The existence of a sustainable state is not completely obvious. Clearly such a state must be of the form $X^i = (x^i, \ldots, x^i, \bar{x}^i)$ with an invariant optimal harvesting policy: harvest x^i and sow exactly the same area in order to keep an invariant configuration. It is also clear that we must have $\bar{x}^i = 0$ since otherwise a policy that harvests a little more at time $t = 0$ and x^i in all other periods would provide a greater benefit contradicting optimality. Since the area constraint imposes $\sum_{i \in I} n_i x^i = S$ we are left with only $k-1$ degrees of freedom. For the rest of this paper we denote $\sigma_i = b^{n_i}/(1-b^{n_i})$ and without loss of generality we assume that the species are ordered in such a way that $\sigma_1 U_1'(0) \geq \sigma_2 U_2'(0) \geq \cdots \geq \sigma_k U_k'(0)$.

Let $\mathbb{X}^* \in \Delta^0$ be a state of the previous form with $\bar{x}^i = 0$ and $x^i = x^{*i}$ where x^* is the unique optimal solution to the strictly convex program

$$(S) \quad \begin{cases} \max & \sum_{i \in I} n_i \sigma_i U_i(x^i) \\ \text{s.t.} & x^i \geq 0 \text{ and } \sum_{i \in I} n_i x^i = S. \end{cases}$$

Denote $I^* = \{i \in I : x^{*i} > 0\}$ the species which are present in $\mathbb{X}^*$ and let r be the Lagrange multiplier associated to the area constraint $\sum_{i \in I} n_i x^i = S$, so that the optimal solution is characterized by $\sigma_i U_i'(x^{*i}) = r$ for $i \in I^*$ and $\sigma_j U_j'(0) \leq r$ for $j \notin I^*$. The ordering of the species and the strict concavity of U_i then imply that whenever $x^{*i} > 0$ we must also have $x^{*j} > 0$ for all $j < i$, so that $I^* = \{1, \ldots, i^*\}$ for some index i^*. This leads to a constructive method for finding the optimal solution of (S) in which the values of $x^1, x^2, x^3, \ldots$ are increased sequentially:

- increase x^1 as much as possible until $\sigma_1 U_1'(x^1)$ decreases to the value $\sigma_2 U_2'(0)$
- continue increasing x^1 and x^2 simultaneously preserving the equality $\sigma_1 U_1'(x^1) = \sigma_2 U_2'(x^2)$ until this common value decreases to the level $\sigma_3 U_3'(0)$.
- continue increasing x^1, x^2, x^3 keeping the equality $\sigma_1 U_1'(x^1) = \sigma_2 U_2'(x^2) = \sigma_3 U_3'(x^3)$ until this common value hits the level $\sigma_4 U_4'(0)$.
- continue this procedure with $x^4, x^5, \ldots, x^{i^*}$ stopping as soon as $\sum_{i=1}^{i^*} n_i x^i = S$.

The point found by this procedure satisfies the optimality conditions for (S) and it is therefore the unique optimal solution. More interestingly, we have

Proposition 2.3.2. *The state $\mathbb{X}^*$ defined above is the unique sustainable state.*

Proof. Let us first show that the stationary trajectory $x_t^i = x^{*i}$, $\bar{x}_t^i = 0$, $u_t^i = x^{*i}$, is optimal for $P(\mathbb{X}^*)$. To this end it suffices to consider the Lagrangian

$$
\text{(2.4)} \qquad L = \sum_{i\in I} \left\{ \sum_{t=0}^{\infty} b^t U_i(u_t^i) + \sum_{t=0}^{\infty} \mu_t^i u_t^i + \sum_{t=1}^{\infty} \bar{\lambda}_t^i \bar{x}_t^i + \sum_{t=n_i}^{\infty} \lambda_t^i x_t^i \right\} + \sum_{i\in I} \sum_{t=0}^{\infty} \alpha_t^i (\bar{x}_t^i + x_t^i - u_t^i - \bar{x}_{t+1}^i) + \sum_{t=0}^{\infty} [\theta_t \sum_{i\in I} (u_t^i - x_{t+n_i}^i)].
$$

together with the following set of ℓ^1-multipliers

$$
\begin{cases}
\mu_t^i = 0 \\
\theta_t = b^t r \\
\alpha_t^i = b^t \big[r + U_i'(x^{*i}) \big] \\
\lambda_t^i = b^t \big[\frac{r}{\sigma_i} - U_i'(x^{*i}) \big] \\
\bar{\lambda}_t^i = \alpha_t^i (1-b)/b
\end{cases}
$$

where r is as before the Lagrange multiplier corresponding to the area constraint in (S). A routine verification shows that $\nabla L = 0$ and we have complementary slackness so the proposed trajectory is a stationary point for $P(\mathbb{X}^*)$, hence an optimal solution. This proves that $\mathbb{X}^*$ is sustainable.

To prove the converse let $\mathbb{X}$ be a sustainable state with $X^i = (x^i, \ldots, x^i, 0)$. We claim that when $x^i > 0$ then $\sigma_i U_i'(x^i) \geq \sigma_j U_j'(x^j)$ for all $j \in I$. Indeed, let us perturb the optimal harvesting policy as follows: at time $t = 0$ we sow $x^i - \epsilon$ and $x^j + \epsilon$ instead of x^i and x^j, while in all subsequent periods we harvest all mature trees and sow the harvested areas with the same species they had. The benefit V_ϵ derived from this perturbed policy must be less than the value V obtained with the optimal one, which gives

$$
V_\epsilon - V = \tfrac{b^{n_i}}{1-b^{n_i}} [U_i(x^i - \epsilon) - U_i(x^i)] + \tfrac{b^{n_j}}{1-b^{n_j}} [U_j(x^j + \epsilon) - U_j(x^j)] \leq 0.
$$

Dividing by ϵ and letting $\epsilon \to 0$ we deduce $\sigma_i U_i'(x^i) \geq \sigma_j U_j'(x^j)$ as claimed. Setting $I^* = \{i : x^i > 0\}$ it follows that $\sigma_i U'(x^i)$ is constant for $i \in I^*$ and larger than the value of this expression for $i \notin I^*$. This implies that the vector $(x^i)_{i\in I}$ is an optimal solution to (S) so that $x^i = x^{*i}$ completing the proof. ■

2.3.2 Greedy periodic cycles

For a one-species forest it was proved in [30] that after a finite time every optimal trajectory becomes *greedy* in the sense that all mature trees are harvested, and then the evolution becomes periodic. For the multiple-species case this notion of greedy trajectory is incomplete since it does not specify a sowing policy. A typical example of a *greedy* evolution for 2 species with

$n_1 = 3$ and $n_2 = 2$ would be

$$\begin{pmatrix} \bar{x}_0^1 \\ x_0^1 \\ x_1^1 \\ x_2^1 \end{pmatrix} \begin{pmatrix} \bar{x}_0^2 \\ x_0^2 \\ x_1^2 \end{pmatrix} \to \begin{pmatrix} 0 \\ x_1^1 \\ x_2^1 \\ s \end{pmatrix} \begin{pmatrix} 0 \\ x_1^2 \\ t \end{pmatrix}$$

with $s + t = \bar{x}_0^1 + x_0^1 + \bar{x}_0^2 + x_0^2$. We will refer to a *greedy periodic strategy* when the new seedlings preserve the area just harvested for each species, *i.e.* $s = \bar{x}_0^1 + x_0^1$ and $t = \bar{x}_0^2 + x_0^2$.

Definition 2.3.3. A feasible trajectory $(\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i)_{i\in I}$ is called *greedy* if $\bar{x}_t^i = 0$. It will be called a *greedy periodic cycle (GPC)* if in addition $x_{t+n_i}^i = u_t^i$. We denote Δ^g the set of initial states $\mathbb{X}_0 \in \Delta^0$ for which there exists an optimal trajectory which is greedy, and Δ^p those having an optimal trajectory that is a GPC.

The presence of a discount factor b implies an advantage for earlier harvests suggesting the convenience of greedy strategies. However this is balanced by the concavity of U_i that gives an advantage to harvesting the same amount at each stage. Hence, a transient period may exist in which some trees are kept beyond maturity in order to transfer area between different age-classes and species. In §2.4 we will describe two situations in which these area transfers occur only during an initial phase after which every optimal trajectory becomes greedy. On the other hand, it is worth mentioning that since the optimal harvests are uniquely determined, the area balance implies that an optimal trajectory issued from an initial state $\mathbb{X}_0 \in \Delta^g$ must be unique, namely a greedy one, and therefore Δ^g is forward invariant through optimal policies: $\mathbb{X}_t \in \Delta^g \Rightarrow \mathbb{X}_{t+1} \in \Delta^g$. In the case of a GPC the dynamics of all the species are uncoupled producing cyclic trajectories of period n_i for species i, so that the sequence of states $(\mathbb{X}_t)_{t\in\mathbb{N}}$ is cyclic with period equal to the least common multiple N of the maturity ages $n_1, \ldots, n_k$.

Clearly the sustainable state $\mathbb{X}^*$ yields an optimal GPC so that $\mathbb{X}^* \in \Delta^p$. In §2.3.3 we will show that under appropriate conditions this is in fact the only point in Δ^p. Let us first characterize this set.

Theorem 2.3.4. *Let $\mathbb{X}_0 \in \Delta^0$ and consider the periodic sequences $(x_t^i)_{t\in\mathbb{N}}$ built from $\mathbb{X}_0$ with $x_{t+n_i}^i = x_t^i$. Then $\mathbb{X}_0 \in \Delta^p$ iff for all $i, j \in I$ and $t \in \mathbb{N}$ we have*[1]

$$\begin{aligned} &a) \quad U_i'(x_t^i) \geq b\, U_i'(x_{t+1}^i) \\ &b) \quad x_t^i > 0 \quad \Rightarrow \quad \sigma_i U_i'(x_t^i) \geq b^{n_j} \left[\sigma_i U_i'(x_{t+n_j}^i) + U_j'(x_t^j)\right]. \end{aligned} \tag{2.5}$$

In the proof we will exploit the following useful consequence of the latter condition.

[1]Notice that it suffices to check condition $a)$ for $t = 0, 1, \ldots, n_i - 1$ while for condition $b)$ it suffices to check for $t = 0, 1, \ldots, n_{ij} - 1$ where n_{ij} is the least common multiple of the maturity ages n_i and n_j.

Lemma 2.3.5. *If condition* (2.5)b *holds then*

$$x_t^i > 0 \quad \Rightarrow \quad \sigma_i U_i'(x_t^i) \geq \sigma_j U_j'(x_t^j). \tag{2.6}$$

Proof. Let us observe that if the inequality in (2.5)b holds at a certain time t then it also holds at time $t+n_j$. Indeed, if this was not the case we would have $x_{t+n_j}^i = 0$ and the inequalities

$$\begin{aligned} \sigma_i U_i'(x_t^i) &\geq b^{n_j}[\sigma_i U_i'(x_{t+n_j}^i) + U_j'(x_t^j)] \\ b^{n_j}[\sigma_i U_i'(x_{t+2n_j}^i) + U_j'(x_t^j)] &> \sigma_i U_i'(x_{t+n_j}^i) \end{aligned}$$

added together would imply

$$\sigma_i[U_i'(x_t^i) + b^{n_j} U_i'(x_{t+2n_j}^i)] > \sigma_i(1+b^{n_j})U_i'(0)$$

contradicting the fact that $U_i'(\cdot)$ is decreasing.

Now, if $x_t^i > 0$ then it follows inductively that (2.5)b holds for all integers of the form $t+\ell n_j$ with $\ell \in I\!N$. Writing down these inequalities for $\ell = 0, 1, \ldots, n_i - 1$ we get

$$\begin{aligned} \sigma_i U_i'(x_t^i) &\geq b^{n_j}[\sigma_i U_i'(x_{t+n_j}^i) + U_j'(x_t^j)] \\ b^{n_j}\sigma_i U_i'(x_{t+n_j}^i) &\geq b^{2n_j}[\sigma_i U_i'(x_{t+2n_j}^i) + U_j'(x_t^j)] \\ &\vdots \\ b^{(n_i-1)n_j}\sigma_i U_i'(x_{t+(n_i-1)n_j}^i) &\geq b^{n_i n_j}[\sigma_i U_i'(x_{t+n_i n_j}^i) + U_j'(x_t^j)] \end{aligned}$$

which added together yield

$$\sigma_i U_i'(x_t^i) \geq b^{n_i n_j}\sigma_i U_i'(x_t^i) + (b^{n_j} + \cdots + b^{n_i n_j})U_j'(x_t^j).$$

This inequality may be equivalently rewritten as

$$(1 - b^{n_i n_j})\sigma_i U_i'(x_t^i) \geq \tfrac{1-b^{n_i n_j}}{1-b^{n_j}} b^{n_j} U_j'(x_t^j)$$

which readily gives $\sigma_i U_i'(x_t^i) \geq \sigma_j U_j'(x_t^j)$. ∎

Proof of Theorem 2.3.4. To prove the necessity of condition (2.5) take $\mathbb{X}_0 \in \Delta^p$ such that the periodic sequences $(x_t^i)_{t\in I\!N}$ are optimal for $P(\mathbb{X}_0)$.

If $x_t^i = 0$ property (2.5)a is obvious since U_i' is decreasing. Suppose next that $x_t^i > 0$ and perturb the GPC by harvesting $x_t^i - \epsilon$ at stage t keeping ϵ for the next period as over-mature trees, and then harvest $x_{t+1}^i + \epsilon$ at stage $t+1$ after which we continue with a GPC. This modification must give a smaller benefit than the optimal GPC, from which we get

$$b^t \tfrac{1}{1-b^{n_i}}[U_i(x_t^i - \epsilon) - U_i(x_t^i)] + b^{t+1}\tfrac{1}{1-b^{n_i}}[U_i(x_{t+1}^i + \epsilon) - U_i(x_{t+1})] \leq 0$$

which divided by ϵ and letting it to 0 yields (2.5)a.

A similar argument proves (2.5)b. The idea is now as follows: at stage t we sow $x_t^i - \epsilon$ instead of x_t^i on species i and transfer this ϵ to species j. Then, at stage $t + n_j$ we harvest this ϵ from species j and return it to species i modifying the sowing at that stage. In all the other stages we use a greedy periodic strategy. Noting that $x_{t+n_j}^j = x_t^j$, the difference of benefit is given by

$$\begin{aligned} b^t \tfrac{b^{n_i}}{1-b^{n_i}}[U_i(x_t^i - \epsilon) - U_i(x_t^i)] + b^{t+n_j} \tfrac{b^{n_i}}{1-b^{n_i}}[U_i(x_{t+n_j}^i + \epsilon) - U_i(x_{t+n_j}^i)] \\ + b^{t+n_j}[U_j(x_t^j + \epsilon) - U_j(x_t^j)] \quad \leq \quad 0 \end{aligned}$$

and the conclusion follows as before dividing by ϵ and letting it to 0.

Let us establish next the sufficiency. Take $\mathbb{X}_0$ satisfying (2.5) and consider the corresponding GPC. In order to prove its optimality it suffices to check that it is a stationary point for the Lagrangian (2.4) with the following multipliers

$$\begin{cases} \mu_t^i = 0 \\ \theta_t = b^t \sigma_1 U_1'(x_t^1) \\ \lambda_t^i = b^t \{\sigma_1[\frac{1}{b^{n_i}} U_1'(x_{t-n_i}^1) - U_1'(x_t^1)] - U_i'(x_t^i)\} \\ \alpha_t^i = \theta_t + b^t U_i'(x_t^i) \\ \bar{\lambda}_t^i = \alpha_{t-1}^i - \alpha_t^i. \end{cases}$$

All these multipliers are of the form b^t multiplied by some bounded sequence so they belong to ℓ^1, while their non-negativity is evident except for λ_t^i and $\bar{\lambda}_t^i$. The inequality $\bar{\lambda}_t^i \geq 0$ follows directly from (2.5)a. For $\lambda_t^i \geq 0$ we observe that this is assured by condition (2.5)b whenever $x_{t-n_i}^1 > 0$, while when $x_{t-n_i}^1 = 0$ using the monotonicity of U_i' and the fact that $\sigma_1 U_1'(0) \geq \sigma_i U_i'(0)$ we get

$$\sigma_1[\tfrac{1}{b^{n_i}} U_1'(0) - U_1'(x_t^1)] - U_i'(x_t^i) \geq \sigma_1[\tfrac{1}{b^{n_i}} - 1] U_1'(0) - U_i'(x_t^i) \geq U_i'(0) - U_i'(x_t^i) \geq 0$$

which shows that all the multipliers belong to ℓ_+^1.

Verification of stationarity is straightforward and only the verification of the complementary slackness for the constraint $x_t^i \geq 0$ is not obvious. Let us show that $x_t^i > 0 \Rightarrow \lambda_t^i = 0$. Indeed, when $x_t^i > 0$ we know from Lemma 2.3.5 that $\sigma_i U_i'(x_t^i) \geq \sigma_1 U_1'(x_t^1)$. Now, if $x_t^1 > 0$ the same argument yields $\sigma_1 U_1'(x_t^1) \geq \sigma_i U_i'(x_t^i)$, while when $x_t^1 = 0$ we have $\sigma_1 U_1'(x_t^1) = \sigma_1 U_1'(0) \geq \sigma_i U_i'(0) \geq \sigma_i U_i'(x_t^i)$. Hence, in all cases we have $\sigma_i U_i'(x_t^i) = \sigma_1 U_1'(x_t^1)$. A similar reasoning yields $\sigma_i U_i'(x_t^i) = \sigma_1 U_1'(x_{t-n_i}^1)$ which readily implies $\lambda_t^i = 0$. This completes the proof of optimality of the GPC for $P(\mathbb{X}_0)$. ■

Theorem 2.3.4 may also be used to characterize the optimal GPCs for a one-species forest.

Indeed, introducing an auxiliary annual second species with benefit $U_2 \equiv 0$ (note that in the previous proof we did not use strict monotonicity nor strict concavity of the U_i's), we get

Corollary 2.3.6. *For a one-species forest the initial state* $X_0 = (x_{n-1}, \ldots, x_0, 0)$ *produces an optimal GPC if and only if* $U'(x_t) \geq b\, U'(x_{t+1})$ *for all* $t = 0, \ldots, n-1$.

2.3.3 Relation between the GPCs and the sustainable state

We already observed that $\mathbb{X}^* \in \Delta^p$. We will prove that $\mathbb{X}^*$ is in fact the only element in Δ^p whenever the maturity ages of the species which are present in $\mathbb{X}^*$ are relatively primes. Let us define the *support* of a given state $\mathbb{X}_0 \in \Delta^p$ as the set of indices $I(\mathbb{X}_0) = \{i \in I : X^i \neq 0\}$.

Lemma 2.3.7. *If* $\mathbb{X}_0 \in \Delta^p$ *then* $I(\mathbb{X}_0) = \{1, \ldots, i_0\}$ *for some* $i_0 \geq i^*$.

Proof. We observe that if $X^i \equiv 0$ then $X^j \equiv 0$ for all $j > i$, which implies that $I(\mathbb{X}_0) = \{1, \ldots, i_0\}$. This first statement is a direct consequence of Lemma 2.3.5, the strict concavity of the functions U_i, and the order in which species are numbered so that $\sigma_i U_i'(0)$ decreases with i. Now, in order to prove that $i_0 \geq i^*$ we proceed by contradiction assuming that $i^* > i_0$. The area balance

$$\textstyle\sum_{i \in I_0} \sum_{t=0}^{n_i - 1} x_t^i = S = \sum_{i \in I^*} \sum_{t=0}^{n_i - 1} x^{i*}$$

implies that there exist $i \leq i_0$ and $t \in I\!N$ such that $x_t^i > x^{i*}$ and thus

$$\sigma_i U_i'(x_t^i) < \sigma_i U_i'(x^{i*}) = \sigma_j U_j'(x^{j*})\ \forall j \in I^*.$$

In particular, taking $j \in I^* \setminus I_0$ and using Lemma 2.3.5 we reach the contradiction

$$\sigma_i U_i'(x_t^i) \geq \sigma_j U_j'(0) > \sigma_j U_j'(x^{*j}) > \sigma_i U_i'(x_t^i).$$

■

In the sequel we denote $(n_1, \ldots, n_i)$ the greatest common divisor of the integers $n_1, \ldots, n_i$.

Theorem 2.3.8. *If* $(n_1, \ldots, n_{i*}) = 1$ *then* $\Delta^p = \{\mathbb{X}^*\}$. *Otherwise* $(n_1, \ldots, n_{i_0}) > 1$ *for all* $\mathbb{X}_0 \in \Delta^p$.

Proof. By Lemma 2.3.7 it suffices to show that if $\mathbb{X}_0 \in \Delta^p$ is such that $(n_1, \ldots, n_{i_0}) = 1$ then $\mathbb{X}_0 = \mathbb{X}^*$. To prove the latter we shall use the following direct consequence of the Chinese Remainder Theorem: if $v \equiv u \bmod (n, m)$ there exist $\ell, k \in I\!N$ such that $v + \ell\, m = u + k\, n$.

Fix u with $x_u^{i_0} > 0$.

For each $v \equiv u \bmod (n_{i_0}, n_{i_0-1})$ take $\ell, k \in I\!N$ with $v + \ell\, n_{i_0-1} = u + k\, n_{i_0}$ so that (2.6) yields

$$\sigma_{i_0} U'_{i_0}(x_u^{i_0}) \geq \sigma_{i_0-1} U'_{i_0-1}(x_v^{i_0-1}).$$

This implies in turn that $x_v^{i_0-1} > 0$ since otherwise we reach the contradiction

$$\sigma_{i_0} U'_{i_0}(x_u^{i_0}) \geq \sigma_{i_0-1} U'_{i_0-1}(0) \geq \sigma_{i_0} U'_{i_0}(0) > \sigma_{i_0} U'_{i_0}(x_u^{i_0}),$$

and therefore using (2.6) once again we deduce

$$x_v^{i_0-1} > 0 \quad \text{and} \quad \sigma_{i_0-1} U'_{i_0-1}(x_v^{i_0-1}) = \sigma_{i_0} U'_{i_0}(x_u^{i_0}). \tag{2.7}$$

Now, if $w \equiv u \bmod (n_{i_0}, n_{i_0-1}, n_{i_0-2})$ we may take $\ell, k \in I\!N$ with $w + \ell\, n_{i_0-2} = u + k\,(n_{i_0}, n_{i_0-1})$. The integer $v = u + k\,(n_{i_0}, n_{i_0-1})$ is obviously congruent with u modulo (n_{i_0}, n_{i_0-1}) so we have (2.7) and then using (2.6) we get

$$\sigma_{i_0-1} U'_{i_0-1}(x_v^{i_0-1}) \geq \sigma_{i_0-2} U'_{i_0-2}(x_w^{i_0-2}).$$

As in the previous argument we may not have $x_w^{i_0-2} = 0$ so we deduce

$$x_w^{i_0-2} > 0 \quad \text{and} \quad \sigma_{i_0-2} U'_{i_0-2}(x_w^{i_0-2}) = \sigma_{i_0} U'_{i_0}(x_u^{i_0}).$$

Proceeding inductively we conclude that for all $t \equiv u \bmod (n_{i_0}, \ldots, n_1)$ we have

$$x_t^1 > 0 \quad \text{and} \quad \sigma_1 U'_1(x_t^1) = \sigma_{i_0} U'_{i_0}(x_u^{i_0})$$

and since $(n_1, \ldots, n_{i_0}) = 1$ it follows that $x_t^1 = x^1$ is constant for all $t \in I\!N$. To establish the theorem it remains to prove that

$$\sigma_i U'_i(x_t^i) = \sigma_1 U'_1(x^1) \quad \forall i \in I(\mathbb{X}_0), t \in I\!N \tag{2.8}$$

since this clearly forces $\mathbb{X}_0$ to be the sustainable state $\mathbb{X}^*$. We already know by Lemma 2.3.5 that (2.8) holds when $x_t^i > 0$. We finish the proof by noting that for each $i \in I(\mathbb{X}_0)$ there exists at least one $t \in I\!N$ with $x_t^i > 0$, and then there cannot be an element $x_u^i = 0$ because that would imply

$$\sigma_1 U'_1(x^1) \geq \sigma_i U'_i(0) > \sigma_i U'_i(x_t^i) = \sigma_1 U'_1(x^1).$$

■

2.4 Convergence of optimal trajectories

We turn next to the study of the long run behavior of the optimal harvesting policies. The previous sections described some special states from which the optimal trajectory is either invariant or periodic. We claim that such behavior is typical in the sense that an optimally managed forest converges either to the sustainable state or to an optimal GPC. To prove this *global attractor property* we first establish conditions under which all optimal trajectories become greedy after a finite time, and then we rely on a suitable Lyapunov function to analyze their asymptotic behavior. Finally we show that the result holds not only for greedy trajectories but for every optimal harvesting policy.

2.4.1 Optimal trajectories become greedy

We start by giving conditions under which every optimal strategy becomes greedy after finitely many steps. Throughout we let $\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i$ be an optimal trajectory for $P(\mathbb{X}_0)$, while $X_t^i = (x_{t+n_i-1}^i, \ldots, x_t^i, \bar{x}_t^i)$ stands for the corresponding state of the i-th species in the forest at time t and $\mathbb{X}_t = (X_t^1, \ldots, X_t^k)$. Our first result concerns the case of an annual alternative use of the land such as agriculture or rent.

Proposition 2.4.1. *If there is a maturity age $n_d = 1$, then $\mathbb{X}_t \in \Delta^g$ for all $t \geq 2\bar{n} - 1$ where $\bar{n} = \max_{i \in I} n_i$.*

Proof. Since $n_d = 1$ there is no gain in postponing the harvest of mature areas for that species: harvesting all of $\bar{x}_t^d$ and resowing it immediately with the same species provides a feasible trajectory that yields a strictly larger benefit which contradicts optimality. Hence $\bar{x}_t^d = 0$ for $t \geq 1$.

For the species $i \neq d$ the argument is a bit more elaborated.

First note that in each interval of length n_i such as $p+1, \ldots, p+n_i$ there is at least one $\bar{x}_t^i = 0$. Indeed, if this was not the case then at time p we could harvest a small additional area $\epsilon > 0$ and resow it immediately as species i, modifying the trajectory as $\bar{x}_t^i - \epsilon$ for $t = p+1, \ldots, p+n_i$ and $x_{p+n_i}^i + \epsilon$, after which we rejoin the original optimal strategy. This modified trajectory would increase the benefit by an amount $b^p[U_i(u_p^i + \epsilon) - U_i(u_p^i)] > 0$ contradicting optimality.

Next observe that for $t \geq n_i$ we have $\bar{x}_t^i = 0 \Rightarrow \bar{x}_{t+1}^i = 0$. To see this we proceed again by contradiction: if $\bar{x}_t^i = 0 < \bar{x}_{t+1}^i$ then $u_t^i < \bar{x}_t^i + x_t^i = x_t^i$ which means that at stage t we do not harvest all the available trees of species i. If we backtrack to stage $t - n_i$ when x_t^i was sown, we could have taken out a small area $\epsilon > 0$ and sow it as species d making an additional benefit in the next period of $b^{t-n_i+1}[U_d(u_{t-n_i+1}^d + \epsilon) - U_d(u_{t-n_i+1}^d)] > 0$, after which this ϵ is returned to species i so that at stage $t+1$ the trees reaching maturity $x_{t+1}^i + \epsilon$ compensate the loss of over-mature trees $\bar{x}_{t+1}^i - \epsilon$. This trajectory allows to harvest the same areas as in the original strategy, except at stage $t - n_i + 1$ where we make an extra benefit which contradicts optimality.

Combining the previous properties we may conclude: in the interval $n_i, \ldots, 2n_i - 1$ there exists at least one t such that $\bar{x}_t^i = 0$, condition that must hold thereafter. ■

Remark 2.4.2. The previous result is sharp since there are examples with $\bar{x}_{2n_i-2}^i > 0$ such as a two species forest initially composed only of young trees: $X_0^1 = (S, 0, \ldots, 0)$ and $X_0^2 = (0, 0)$. We must wait already n_1 periods to have available trees to harvest, and then the optimal strategy may consist in harvesting and sowing only a fraction of them, keeping the rest as over-mature trees in order to reshape the forest with a more balanced age-class distribution. This process may take up to $n_1 - 1$ additional periods to reach the regime $\bar{x}_t^1 \equiv 0$.

When $n_i > 1$ for all i we could not prove in general that the optimal trajectories become greedy, but we can give the following sufficient condition, which holds in particular when the benefit functions U_i are "almost" linear or when either b or S are small.

Proposition 2.4.3. *If $U_i'(S) \geq b\, U_i'(0)$ for all i, then $\mathbb{X}_t \in \Delta^g$ for all $t \geq 1$.*

Proof. We must show that $\bar{x}_t^i = 0$ for all $i \in I$ and $t \geq 1$. Since all the cases are symmetric we just prove it for $i=1$. The argument in the previous proof shows that in any interval of length n_1 there is at least one $\bar{x}_t^1 = 0$ so we may find t as large as we want with $\bar{x}_t^1 = 0$. The conclusion will follow by backward induction if we show that $\bar{x}_{t+1}^1 = 0 \Rightarrow \bar{x}_t^1 = 0$ for $t \geq 1$.

To prove the latter we proceed by contradiction: suppose $\bar{x}_t^1 > \bar{x}_{t+1}^1 = 0$ so that $u_t^1 = \bar{x}_t^1 + x_t^1 > 0$ and then there is at least one $i \in I$ such that $x_{t+n_i}^i > 0$. We take $\epsilon = \min\{\bar{x}_t^1, x_{t+n_i}^i\} > 0$ and consider the perturbed trajectory that anticipates the harvest at time t as $u_{t-1}^1 + \epsilon, u_t^1 - \epsilon, \bar{x}_t^1 - \epsilon$ and the sowing of species i as $x_{t+n_i-1}^i + \epsilon, x_{t+n_i}^i - \epsilon, \bar{x}_{t+n_i}^i + \epsilon$, with all other variables untouched. This trajectory is still feasible and strict concavity of U_1 implies that the benefit difference is

$$\begin{aligned} V_\epsilon - V &= b^{t-1}[U_1(u_{t-1}^1 + \epsilon) + b\, U_1(u_t^1 - \epsilon) - U_1(u_{t-1}^1) - b\, U_1(u_t^1)] \\ &> b^{t-1}\epsilon[U_1'(u_{t-1}^1 + \epsilon) - b\, U_1'(u_t^1 - \epsilon)] \end{aligned}$$

so that $V_\epsilon - V > b^{t-1}\epsilon[U_1'(S) - b\, U_1'(0)] \geq 0$ contradicting optimality. ∎

2.4.2 Lyapunov function

In §2.4.1 we described conditions under which an optimally managed forest attains the set Δ^g in finite time and remains in this set thereafter. This does not mean that the optimal policy leads to a GPC nor to the sustainable state, since we have not yet proved that the sown areas coincide with the harvests. In order to address this issue we introduce the Lyapunov function $\Phi : \Delta^0 \to I\!R$ given by

$$\Phi(\mathbb{X}_0) = G(\mathbb{X}_0) - \sum_{i \in I} \sum_{t=0}^{n_i - 2} \frac{b^t - b^{n_i - 1}}{1 - b^{n_i}} U_i(x_t^i) \tag{2.9}$$

where $G(\mathbb{X}_0)$ is the optimal benefit obtained from state $\mathbb{X}_0$ by using a greedy policy

$$\begin{aligned} G(\mathbb{X}_0) \;=\; & \max \;\; \sum_{i \in I} \sum_{t=0}^{\infty} b^t U_i(x_t^i) \\ & \text{s.t. } \boldsymbol{x}^i \in \ell_+^\infty \\ & \sum_{i \in I} x_{t+n_i}^i = \sum_{i \in I} x_t^i. \end{aligned}$$

Naturally, when restricted to states in Δ^g the latter coincides with the optimal value of $P(\mathbb{X}_0)$. The clue for the subsequent asymptotic analysis is the following property.

Theorem 2.4.4. *Let N be the least common multiple of $n_1, \ldots, n_k$. Then for all $\mathbb{X}_0 \in \Delta^g$ we have*

$$\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0)$$

with strict inequality unless $\mathbb{X}_0 \in \Delta^p$.

Proof. To simplify the notation set $U_t^i = U_i(x_t^i)$, $G_t = G(\mathbb{X}_t)$, and denote

$$P_t = \textstyle\sum_{i\in I} \frac{1}{1-b^{n_i}} \sum_{j=t}^{t+n_i-1} b^{j-t} U_j^i$$

the benefit of a GPC started from state $\mathbb{X}_t$. Since G_t is the optimal greedy benefit we have $G_t \geq P_t$ which can be written as $(1-b^N)G_t \geq (1-b^N)P_t$ and then

$$G_t \geq b^N G_t + \textstyle\sum_{i\in I} \frac{1-b^N}{1-b^{n_i}} \sum_{j=t}^{t+n_i-1} b^{j-t} U_j^i.$$

Now, Bellman's principle of dynamic programming gives

$$\begin{aligned} G_0 &= \textstyle\sum_{i\in I}\sum_{j=0}^{t-1} b^j U_j^i + b^t G_t \\ G_t &= \textstyle\sum_{i\in I}\sum_{j=t}^{N-1} b^{j-t} U_j^i + b^{N-t} G_N \end{aligned}$$

which plugged into the previous inequality yields

$$b^{N-t}G_N + \sum_{i\in I}\sum_{j=t}^{N-1} b^{j-t}U_j^i \geq b^{N-t}G_0 - \sum_{i\in I}\sum_{j=0}^{t-1} b^{N-t+j}U_j^i + \textstyle\sum_{i\in I} \frac{1-b^N}{1-b^{n_i}} \displaystyle\sum_{j=t}^{t+n_i-1} b^{j-t}U_j^i.$$

Adding up these equations for $t = 0, 1, \ldots, N-1$ we get

$$\begin{aligned} \frac{b(1-b^N)}{1-b} G_N + \sum_{t=0}^{N-1}\sum_{i\in I}\sum_{j=t}^{N-1} b^{j-t}U_j^i \;&\geq\; \frac{b(1-b^N)}{1-b} G_0 - \sum_{t=0}^{N-1}\sum_{i\in I}\sum_{j=0}^{t-1} b^{N-t+j}U_j^i \\ &+ \sum_{t=0}^{N-1}\sum_{i\in I} \frac{1-b^N}{1-b^{n_i}} \sum_{j=t}^{t+n_i-1} b^{j-t}U_j^i \end{aligned} \tag{2.10}$$

and exchanging the order in all these multiples summations we deduce

$$\begin{aligned} \frac{b(1-b^N)}{1-b} G_N + \sum_{j=0}^{N-1} \frac{1-b^{j+1}}{1-b} \sum_{i\in I} U_j^i \;&\geq\; \frac{b(1-b^N)}{1-b} G_0 - \sum_{j=0}^{N-1} \frac{b^{j+1}-b^N}{1-b} \sum_{i\in I} U_j^i \\ &+ \sum_{i\in I} \frac{1-b^N}{1-b^{n_i}} \left[\sum_{j=0}^{n_i-2} \frac{1-b^{j+1}}{1-b} U_j^i + \sum_{j=n_i-1}^{N-1} \frac{1-b^{n_i}}{1-b} U_j^i + \sum_{j=N}^{N+n_i-2} \frac{b^{j-N+1}-b^{n_i}}{1-b} U_j^i \right]. \end{aligned}$$

The two sums on the first line may be combined and factored by the term $\frac{1-b^N}{1-b}$ which may then be dropped throughout in order to deduce

$$b\,G_N + \sum_{j=0}^{N-1}\sum_{i\in I} U_j^i \;\geq\; b\,G_0 + \sum_{i\in I} \left[\sum_{j=0}^{n_i-2} \frac{1-b^{j+1}}{1-b^{n_i}} U_j^i + \sum_{j=n_i-1}^{N-1} \frac{1-b^{n_i}}{1-b^{n_i}} U_j^i + \sum_{j=N}^{N+n_i-2} \frac{b^{j-N+1}-b^{n_i}}{1-b^{n_i}} U_j^i \right].$$

We may now change the order of summation of the first sum, transfer it to the rhs and cancel out the terms in order to get

$$b\,G_N \geq b\,G_0 + \sum_{i\in I}\left[\sum_{j=0}^{n_i-2} \tfrac{b^{n_i}-b^{j+1}}{1-b^{n_i}} U_j^i + \sum_{j=N}^{N+n_i-2} \tfrac{b^{j-N+1}-b^{n_i}}{1-b^{n_i}} U_j^i\right]$$

so that dividing by b and rearranging terms we finally deduce $\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0)$. When $\mathbb{X}_0 \notin \Delta^p$ we have $G_0 > P_0$ and the inequality (2.10) is strict so that we get $\Phi(\mathbb{X}_N) > \Phi(\mathbb{X}_0)$. ∎

Since Δ^g is forward invariant under an optimal strategy with $\mathbb{X}_t \in \Delta^g \Rightarrow \mathbb{X}_{t+1} \in \Delta^g$, it follows that an optimal sequence $\mathbb{X}_t$ becomes an "N-step" uphill strategy for the Lyapunov function Φ as soon as it enters the set Δ^g. This property will be used to study the asymptotic behavior of the optimal trajectories. Strictly speaking Φ is not a Lyapunov function since it does not increase at every stage. We could recover a standard Lyapunov function by considering the sum or the maximum over N consecutive periods, however this would make the arguments unnecessarily obscure.

Theorem 2.4.4 is very general and it holds *under no assumption on the utility functions* U_i. However it is interesting to notice that in our setting where the U_i's are strictly concave and increasing then Φ is also strictly concave and attains its maximum at the sustainable state.

Proposition 2.4.5. $\Phi : \Delta^0 \to I\!R$ *is strictly concave and attains its maximum at* $\mathbb{X}^*$.

Proof. Let us prove first that Φ is strictly concave. The initial terms in the sums defining $G(\mathbb{X}_0)$ are fixed by the initial conditions so that introducing the function

$$\Psi(\mathbb{X}_0) = \textstyle\sum_{i\in I} \sum_{t=0}^{n_i-1} (\frac{1}{b} - b^t)\sigma_i U_i(x_t^i)$$

we may equivalently express Φ as follows

$$\begin{aligned}\Phi(\mathbb{X}_0) \quad = \quad & \Psi(\mathbb{X}_0) + \max \left[\textstyle\sum_{i\in I} \sum_{t=n_i}^{\infty} b^t U(x_t^i)\right] \\ & \text{s.t. } \boldsymbol{x}^i \in \ell_+^\infty \\ & \textstyle\sum_{i\in I} x_{t+n_i}^i = \sum_{i\in I} x_t^i.\end{aligned} \tag{2.11}$$

The result follows by noting that Ψ is strictly concave on Δ^0, while the maximum is a concave function of $\mathbb{X}_0$ which appears as the rhs of a linearly constrained concave problem.

We prove next that the maximum is attained at $\mathbb{X}^*$. Indeed, for the initial state $\mathbb{X}^*$ the maximum in (2.11) is attained by $x_t^i = x^{*i}$, which is a stationary point for the Lagrangian (compare with (2.4))

$$\mathcal{L} = \textstyle\sum_{i\in I} \sum_{t=n_i}^{\infty} [b^t U_i(x_t^i) + \lambda_t^i x_t^i] + \sum_{t=0}^{\infty} \theta_t (\sum_{i\in I} x_t^i - \sum_{i\in I} x_{t+n_i}^i) \tag{2.12}$$

with the following multipliers

$$\begin{cases} \theta_t = b^t r \\ \lambda_t^i = b^t\big[\frac{r}{\sigma_i} - U_i'(x^{*i})\big] \end{cases}$$

where r is the Lagrange multiplier associated with the area constraint in the program (S) that characterizes the sustainable state $\mathbb{X}^*$ (see §2.3.1). Sub-differential calculus implies that the max function in (2.11) admits the super-gradient $(Y^1, \ldots, Y^k)$ with $Y^i = (\theta_{n_i-1}, \ldots, \theta_0, 0)$, and adding $\nabla\Psi(\mathbb{X}^*)$ we get a super-gradient $\mathbb{Y}^* \in \partial\Phi(X^*)$ given by

$$\mathbb{Y}^{*i} = \begin{cases} \frac{r}{b}(1, \ldots, 1, 0) & i \in I^* \\ \frac{\sigma_i}{b}U_i'(0)(1, \ldots, 1, 0) + \big[r - \sigma_i U_i'(0)\big](b^{n_i-1}, \ldots, b, 1, 0) & i \notin I^*. \end{cases}$$

To conclude we notice that for all $\mathbb{X}_0 \in \Delta^0$ we have

$$\begin{aligned} \langle \mathbb{Y}^*, \mathbb{X}_0 - \mathbb{X}^* \rangle &= \tfrac{r}{b}\textstyle\sum_{i \le i^*}\sum_{t=0}^{n_i-1}(x_t^i - x^{*i}) + \sum_{i>i^*}\sum_{t=0}^{n_i-1}[(\frac{1}{b} - b^t)\sigma_i U_i'(0) + b^t r]x_t^i \\ &= \textstyle\sum_{i>i^*}\sum_{t=0}^{n_i-1}(\frac{1}{b} - b^t)[\sigma_i U_i'(0) - r]x_t^i \le 0 \end{aligned}$$

so that $\mathbb{X}^*$ is the unique maximum of Φ over Δ^0. ■

2.4.3 Asymptotic convergence

Since a greedy optimal sequence of states $(\mathbb{X}_t)_{t\in I\!N}$ is an N-step uphill strategy for Φ, it is natural to expect convergence towards the maximum $\mathbb{X}^*$. We prove that this is the case when $\Delta^p = \{\mathbb{X}^*\}$, while the most one can expect in general is convergence to a greedy periodic cycle.

Theorem 2.4.6. *Let $\mathbb{X}_0 \in \Delta$ be such that the optimal trajectory satisfies $\mathbb{X}_t \in \Delta^g$ for some $t \in I\!N$. Then the optimal trajectory converges to a GPC in the sense that*

$$\lim_{t\to\infty} \text{dist}(\mathbb{X}_t, \Delta^p) = 0. \tag{2.13}$$

In particular if $\Delta^p = \{\mathbb{X}^*\}$, as in the case $(n_1, \ldots, n_{i^*}) = 1$, the forest converges to the sustainable state $\lim_{t\to\infty}\mathbb{X}_t = \mathbb{X}^*$.

Proof. It suffices to establish (2.13) for which we show that every accumulation point of $\mathbb{X}_t$ belongs to Δ^p. Suppose by contradiction that we have a sequence $t_j \to \infty$ with $\mathbb{X}_{t_j} \to \mathbb{X}^\infty \notin \Delta^p$, and assume with no loss of generality that $\mathbb{X}_{t_j} \in \Delta^g$ for all j. One of the sets $\{i + qN : q \in I\!N\}$ for $i = 1, \ldots, N$ contains infinitely many t_j's, so that passing to a subsequence we may further assume that $t_j = i + q_j N$ for a fixed i and $q_j \to \infty$.

The set-valued map which assigns to $\mathbb{X}_0 \in \Delta$ the solution set $S(\mathbb{X}_0)$ of $P(\mathbb{X}_0)$ is upper-semi-continuous with respect to the $\sigma(\ell^\infty, \ell^1)$ topology on ℓ^∞. This property combined with the fact that a weak* limit of a greedy trajectory is still greedy, implies that Δ^g is closed so that $\mathbb{X}^\infty \in \Delta^g$. On the other hand, after Definition 2.3.3 we observed that for $\mathbb{X}_0 \in \Delta^g$ the optimal solution is unique so that the map $\mathbb{X}_0 \mapsto S(\mathbb{X}_0)$ is in fact strong-to-weak* continuous from Δ^g to $(\ell^\infty)^{3k}$. It follows that the map which assigns to $\mathbb{X}_0 \in \Delta^g$ the state $\mathbb{X}_N \in \Delta^g$ reached at time N is continuous, and then the same holds for the function $\mathbb{X}_0 \in \Delta^g \mapsto \Phi(\mathbb{X}_N) \in I\!R$.

Now since $\mathbb{X}^\infty \notin \Delta^p$, Theorem 2.4.4 gives $\Phi(\mathbb{X}^\infty_N) > \Phi(\mathbb{X}^\infty)$ and by continuity we may find $\epsilon > 0$ and a neighborhood $\mathcal{V}$ of $\mathbb{X}^\infty$ such that $\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0) + \epsilon$ for all $\mathbb{X}_0 \in \Delta^g \cap \mathcal{V}$. Since $\mathbb{X}_{t_j} \to \mathbb{X}^\infty$ we have $\mathbb{X}_{t_j} \in \Delta^g \cap \mathcal{V}$ for all j large, and then $\Phi(\mathbb{X}_{t_{j+1}}) \geq \Phi(\mathbb{X}_{t_j}) + \epsilon$. This implies $\Phi(\mathbb{X}_{t_j}) \to \infty$ which is impossible since Φ is bounded on Δ^g. This contradiction completes the proof of (2.13). ■

Theorem 2.4.6 remains true, with the same proof, even if one of the utility functions U_i is merely concave non-decreasing. We will use this observation to extend the result from greedy to general optimal trajectories. The idea is to show that an optimal trajectory for $P(\mathbb{X}_0)$ corresponds to a greedy optimal trajectory in an augmented auxiliary problem which includes an additional dummy species interpreted as "bare land". In the auxiliary problem the seedlings are postponed until the time on which they are actually required, transferring the land temporarily to the dummy species in such a way to avoid carrying over-mature trees while keeping the original harvests. More precisely, consider the auxiliary problem

$$\tilde{P}(\mathbb{X}_0) \quad \begin{cases} \text{maximize} & \sum_{t=0}^\infty b^t \big[\sum_{i\in I} U_i(u_t^i) + W(w_t)\big] \\ \text{s.t.} & \sum_{i\in I} x^i_{t+n_i} + w_{t+1} = \sum_{i \in I} u^i_t + w_t \\ & \bar{x}^i_{t+1} = \bar{x}^i_t + x^i_t - u^i_t \\ & \boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i, \boldsymbol{w} \in \ell^\infty_+ \end{cases}$$

which adds an annual species with benefit function $W \equiv 0$. Imposing $w = 0$ gives back $P(\mathbb{X}_0)$ so that $\tilde{P}(\mathbb{X}_0)$ is a relaxation and therefore $V(\tilde{P}(\mathbb{X}_0)) \geq V(P(\mathbb{X}_0))$. In fact both optimal values coincide.

Lemma 2.4.7. $V(\tilde{P}(\mathbb{X}_0)) = V(P(\mathbb{X}_0))$.

Proof. It suffices to find a feasible trajectory for $P(\mathbb{X}_0)$ with value equal to $V(\tilde{P}(\mathbb{X}_0))$. Let $((\boldsymbol{y}^i, \bar{\boldsymbol{y}}^i, \boldsymbol{u}^i)_{i\in I}, \boldsymbol{w})$ be an optimal solution for $\tilde{P}(\mathbb{X}_0)$ and let us build $(\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i)_{i\in I}$ a feasible trajectory for $P(\mathbb{X}_0)$ keeping the harvests unchanged in order to obtain exactly the same benefit. To this end we search for $\mu^i_t \geq 0$ such that the following trajectory for $P(\mathbb{X}_0)$ is feasible

$$(2.14) \qquad \begin{cases} x^i_t = y^i_t + \mu^i_t - \mu^i_{t-1} \\ \bar{x}^i_t = \bar{y}^i_t + \mu^i_{t-1} \\ u^i_t = u^i_t. \end{cases}$$

Set $n = \min_{i\in I} n_i$. Arguing as in the proof of Proposition 2.4.1 it follows that on every interval of length n there must be at least one $w_t = 0$ (otherwise we could transfer a small amount of land from w to the species with maturity age n making an extra benefit). Hence we may find an increasing subsequence $\{t_j\}_{j\in I\!N}$ such that $w_{t_j}=0$ with $t_0=0$ and $|t_{j+1}-t_j|\leq n$. Let us define μ^i inductively for $i=1,2,\ldots,k$ setting $\mu^i_t=0$ for $t<n_i$ while for every $j\geq 1$ we put

$$\begin{cases} \mu^i_{t_j+n_i-1} &= 0 \\ \mu^i_{t+n_i-1} &= \min(w_t-\sum_{\ell=1}^{i-1}\mu^\ell_{t+n_\ell-1}, y^i_{t+n_i}+\mu^i_{t+n_i}) \qquad t_{j-1}<t<t_j \end{cases}$$

where the last recursion is solved backwards starting from $t=t_j-1$ down to $t=t_{j-1}+1$.

Let us show that the trajectory (2.14) is feasible. The inequality $\mu^i_{t+n_i-1}\leq w_t-\sum_{\ell=1}^{i-1}\mu^\ell_{t+n_\ell-1}$ inductively implies $\mu^i\geq 0$ so that $\bar{x}^i\geq 0$, while $x^i\geq 0$ comes from the fact that $\mu^i_{t-1}\leq y^i_t+\mu^i_t$. Since the equality $\bar{x}^i_{t+1}=\bar{x}^i_t+x^i_t-u^i_t$ is checked by direct substitution, it remains to prove $\sum_{i\in I}u^i_t=\sum_{i\in I}x^i_{t+n_i}$ which will follow from the area balance of $\tilde{P}(\mathbb{X}_0)$ if we show that

$$w_t=\sum_{i\in I}\mu^i_{t+n_i-1}. \tag{2.15}$$

We prove the latter by backward induction. This obviously holds for $t=t_j$ since $w_{t_j}=0$ and $\mu^i_{t_j+n_i-1}=0$ for all $i\in I$. Suppose that (2.15) holds at time $t+1$ and let us show that it still holds at time t. We claim that there exists some i such that the minimum in the definition of $\mu^i_{t+n_i-1}$ is attained in the first term, since otherwise we would have

$$w_t>\sum_{i\in I}\mu^i_{t+n_i-1}=\sum_{i\in I}(y^i_{t+n_i}+\mu^i_{t+n_i})=\sum_{i\in I}y^i_{t+n_i}+w_{t+1}$$

which contradicts the area balance in $\tilde{P}(\mathbb{X}_0)$. We deduce $\mu^i_{t+n_i-1}=w_t-\sum_{\ell=1}^{i-1}\mu^\ell_{t+n_\ell-1}$ and then $\mu^\ell_{t+n_\ell-1}=0$ for all $\ell>i$ which yields $w_t=\sum_{\ell\in I}\mu^\ell_{t+n_\ell-1}$ completing the induction step. ■

We may now state our main result on the asymptotic behavior of an optimally managed forest.

Theorem 2.4.8. *Every optimal trajectory of $P(\mathbb{X}_0)$ converges to a GPC in the sense that*

$$\lim_{t\to\infty}\text{dist}(\mathbb{X}_t,\Delta^p)=0. \tag{2.16}$$

In particular if $\Delta^p=\{\mathbb{X}^*\}$, as in the case $(n_1,\ldots,n_{i^*})=1$, the forest converges to the sustainable state $\lim_{t\to\infty}\mathbb{X}_t=\mathbb{X}^*$.

Proof. Let $(\boldsymbol{x}^i,\bar{\boldsymbol{x}}^i,\boldsymbol{u}^i)_{i\in I}$ be an optimal solution for $P(\mathbb{X}_0)$ and consider an equivalent greedy optimal trajectory $((\boldsymbol{y}^i,\bar{\boldsymbol{y}}^i,\boldsymbol{u}^i)_{i\in I},\boldsymbol{w})$ for $\tilde{P}(\mathbb{X}_0)$. We know that $\bar{x}^i_t=0$ for at least one $t\in[n_i,2n_i-1]$. Let t_i be any such t and define $w_t=\sum_{i\in I}\bar{x}^i_{t+n_i}\mathbb{1}_{\{t+n_i\geq t_i\}}$ and

$$\begin{cases} y^i_t=x^i_t\ ;\ \bar{y}^i_t=\bar{x}^i_t & t<t_i \\ y^i_t=u^i_t\ ;\ \bar{y}^i_t=0 & t\geq t_i. \end{cases}$$

This trajectory is feasible for $\tilde{P}(\mathbb{X}_0)$ and becomes greedy for $t \geq \max_{i \in I} t_i$. As we have kept the same harvests u_t^i, the previous lemma implies that the trajectory is also optimal for $P(\mathbb{X}_0)$.

Although the function $W \equiv 0$ is only concave and not strictly concave, the auxiliary problem has still a unique sustainable state in which the dummy species w is not present (problem (S) has a unique solution even if one of the utility functions is merely concave). Moreover it is easy to see the set of states leading to GPCs in the auxiliary problem is given by $\tilde{\Delta}^p = \Delta^p \times \{0\}$. Using the remark after Theorem 2.4.6 we deduce that the modified greedy optimal trajectory converges to $\Delta^p \times \{0\}$. It follows that for the auxiliary optimal path we have $\lim_{t \to \infty} w_t = 0$ so that $\bar{x}_t^i \to 0$ and then $|x_t^i - y_t^i| \to 0$ which allows to conclude (2.16). ■

As a consequence of this result we deduce that every optimal trajectory is *asymptotically* greedy. We already mentioned that for the one species case optimal trajectories become greedy after a finite time, while §2.4.1 described situations in which this occurs for multiple species. In general however, it is an open question whether optimal trajectories become greedy or not.

When $\Delta^p = \{\mathbb{X}^*\}$ the previous result establishes a turnpike property for the optimal harvesting. Such results are known for wide classes of dynamic optimization problems, though most of them assume discount factors close to one since smaller factors may induce complex behaviors including chaos [5, 16, 17, 18, 19, 21, 24, 38]. In contrast our result holds for all values of $b \in (0, 1)$. Moreover, it is also worth mentioning that the turnpike property fails when $\Delta^p \neq \{\mathbb{X}^*\}$ which may occur even for values of b arbitrarily close to 1. This implies that the general turnpike theorems do not apply directly to the optimal harvesting problem.

2.5 Finite time convergence

We have seen that an optimal trajectory approaches a GPC or even the sustainable state. Thus, it is worth investigating if convergence occurs in finite time, in which case $P(\mathbb{X}_0)$ could be reformulated as a finite dimensional problem and solved numerically. Unfortunately, this is not true even for states in Δ^g which admit a greedy optimal trajectory. We restrict the analysis to a case with two species since the general case is too involved. In the sequel we denote $t(n)$ the integer t modulo n.

Proposition 2.5.1. *Suppose* $(n_1, n_2) = 1$ *and* $I^* = \{1, 2\}$. *For each* $\mathbb{X}_0 \in \Delta^g$ *either the optimal trajectory reaches* $\mathbb{X}^*$ *within* $n = \min\{n_1, n_2\}$ *steps or the convergence is asymptotic.*

Proof. The case $n_1 = n_2 = 1$ being trivial we just consider the case $n_1 \neq n_2$. Assume with no loss of generality that $n_1 > n_2$ and suppose that the sustainable state is reached at time T. If $T > n_2$ we consider a perturbed trajectory obtained by moving a small area from X^1 to X^2 in the sowing at stage $T-n_2-1$

$$\tilde{x}_t^1 = \begin{cases} x_t^1-\epsilon & \text{if } t=T-n_2-1+in_1, i \geq 1 \\ x_t^1 & \text{otherwise} \end{cases} \qquad \tilde{x}_t^2 = \begin{cases} x_t^2+\epsilon & \text{if } t=T-1+in_2, i \geq 0 \\ x_t^2 & \text{otherwise.} \end{cases}$$

This new trajectory is feasible so its benefit must be smaller than the optimum. Observing that $x_t^1 = x^{*1}$ and $x_t^2 = x^{*2}$ for all $t \geq T$, this implies

$$b^{T-n_2-1}\sigma_1[U_1(x^{*1}-\epsilon)-U_1(x^{*1})]+b^{T-1}[U_2(x_{T-1}^2+\epsilon)-U_2(x_{T-1}^2)]+b^{T-1}\sigma_2[U_2(x^{*2}+\epsilon)-U_2(x^{*2})] \leq 0.$$

Dividing by $\epsilon > 0$ and letting it to 0 we get $U_2'(x^{*2}) \geq U_2'(x_{T-1}^2)$ which combined with the strict concavity of U_2 implies $x_{T-1}^2 > 0$. But then the previous inequality also holds for $\epsilon < 0$ small, so that dividing again by $\epsilon < 0$ and letting it to 0 we deduce $U_2'(x_{T-1}^2) = U_2'(x^{*2})$. Hence $x_{T-1}^2 = x^{*2}$ and a balance of area at stage $T-1$ yields $x_{T-1}^1 = x^{*1}$ showing that the sustainable state was reached in fact at $T-1$. It follows by backward induction that the sustainable state is reached at a time $T \leq n_2$. ■

We may explicitly characterize the initial conditions for which the corresponding optimal trajectories attain the sustainable state in finite time.

Proposition 2.5.2. *Suppose $n_1 \geq n_2$, $(n_1, n_2) = 1$ and $I^* = \{1, 2\}$. For a given $\mathbb{X}_0 \in \Delta^g$ the optimal trajectory reaches the sustainable $\mathbb{X}^*$ state within n_2 steps if and only if*

a) $x_t^1 + x_t^2 = x^{*1} + x^{*2}$ for $t = 0, \ldots, n_2 - 1$,

b) $x_t^1 = x^{*1}$ for $t = n_2, \ldots, n_1 - 1$.

Proof. Conditions *a)* and *b)* are clearly necessary if we are to reach the sustainable state $\mathbb{X}^*$ within n_2 steps. Conversely we must show that the trajectory $x_t^1 = x^{*1}$ for $t \geq n_1$ and $x_t^2 = x^{*2}$ for $t \geq n_2$ is optimal. To this end we consider the Lagrangian (2.12) and the multipliers $\lambda_t^1 = \lambda_t^2 = 0$ and $\theta_t = b^t r$. It is easy to see that $\nabla L = 0$ and that we have complementary slackness so the proposed trajectory is optimal. ■

We analyze next the finite convergence to a GPC considering separately the cases $n_1 > n_2$ and $n_1 < n_2$.

Proposition 2.5.3. *Suppose $n_1 > n_2$, $(n_1, n_2) = 1$ and $I^* = \{1\}$. For each $\mathbb{X}_0 \in \Delta^g$ either the optimal trajectory reaches Δ^p within n_2 time steps or the convergence is asymptotic.*

Proof. Let us assume that the optimal trajectory reaches a state $\hat{\mathbb{X}} \in \Delta^p$ at time $T > n_2$. If $x_{T-1}^2 > 0$ we consider a perturbed trajectory obtained by moving a small area from species X^2 to X^1 in the sowing at stage $T-n_2-1$ which is then returned to X^2 at time $T-1$

$$\tilde{x}_t^1 = \begin{cases} x_t^1+\epsilon & \text{if } t=T-n_2-1+in_1, i \geq 1 \\ x_t^1-\epsilon & \text{if } t=T-1+in_1, i \geq 1 \\ x_t^1 & \text{otherwise} \end{cases} \qquad \tilde{x}_t^2 = \begin{cases} x_t^2-\epsilon & \text{if } t=T-1 \\ x_t^2 & \text{otherwise.} \end{cases}$$

Notice that the area balance $x^1_{T-1}+x^2_{T-1}=x^1_{T-1+n_1}+x^2_{T-1+n_2}$ together with $x^2_{T-1+n_2}=0$ and $x^2_{T-1}>0$ imply $x^1_{T-1+n_1}>0$. This implies that the alternative trajectory is feasible for all $\epsilon>0$ small enough and therefore its benefit must be smaller than the optimum. Since for $t\geq T$ we have reached the GPC a straightforward computation yields

$$\begin{aligned}[U_2(x^2_{T-1}-\epsilon)-U_2(x^2_{T-1})]+b^{-n_2}\sigma_1[U_1(\hat{x}^1_{n_1-n_2-1}+\epsilon)-U_1(\hat{x}^1_{n_1-n_2-1})]\\ +\sigma_1[U_1(\hat{x}^1_{n_1-1}-\epsilon)-U_1(\hat{x}^1_{n_1-1})] \quad \leq \quad 0.\end{aligned}$$

Dividing by ϵ and letting it to 0 we get $U_2'(x^2_{T-1})\geq\sigma_1\left[\frac{1}{b^{n_2}}U_1'(\hat{x}^1_{n_1-n_2-1})-U_1'(\hat{x}^1_{n_1-1})\right]$ and invoking Theorem 2.3.4 we obtain the contradiction $U_2'(x^2_{T-1})\geq U_2'(0)$.

This contradiction implies $x^2_{T-1}=0$ and then an area balance at stage $T-1$ yields $x^1_{T-1}=\hat{x}^1_{n-1}$ which implies $\mathbb{X}_{T-1}\in\Delta^p$ showing that Δ^p was reached in fact at stage $T-1$. Proceeding inductively we conclude that convergence must occur before stage n_2. ■

Again, it is of interest to characterize the set of initial states whose optimal trajectories reach the set Δ^p in at most n_2 steps.

Proposition 2.5.4. *Suppose $n_1>n_2$, $(n_1,n_2)=1$ and $I^*=\{1\}$. For a given $\mathbb{X}_0\in\Delta^g$ the optimal trajectory reaches Δ^p within n_2 steps if and only if $\mathbb{X}_0'\in\Delta^p$ where*

$$\left\{\begin{array}{lcl} X_0'^1 &=& (x^1_{n_1-1},\ldots,x^1_{n_2},x^1_{n_2-1}+x^2_{n_2-1},\ldots,x^1_0+x^2_0,0)\\ X_0'^2 &=& (0,\ldots,0).\end{array}\right.$$

Proof. It is clear that conditions *a)* and *b)* are necessary in order to attain Δ^p within n_2 steps. Conversely we must show that the trajectory $x^1_t=x'^1_{t(n_1)}$ for $t\geq n_1$ and $x^2_t=0$ for $t\geq n_2$ is optimal, for which it suffices to observe that this trajectory is a stationary point for the Lagrangian (2.12) with respect to the following multipliers

$$\left\{\begin{array}{l}\theta_t=b^t\sigma_1U_1'(x^1_t)\\ \lambda^i_t=\theta_{t-n_i}-\theta_t-b^tU_i'(x^i_t).\end{array}\right.$$

■

The results for the case $n_1<n_2$ are similar but the proofs are more technical.

Proposition 2.5.5. *Suppose $n_1<n_2$, $(n_1,n_2)=1$ and $I^*=\{1\}$. For $\mathbb{X}_0\in\Delta^g$ either the optimal trajectory reaches Δ^p within n_2 steps or the convergence is asymptotic.*

Proof. As in the previous proofs, assume that the optimal trajectory reaches a state $\hat{\mathbb{X}}\in\Delta^p$ at time $T>n_2$. If $x^2_{T-1}>0$ we may consider the following alternative trajectory

$$\tilde{x}^1_t=\left\{\begin{array}{ll}x^1_t+\epsilon & \text{if } t=T+n_1-n_2-1\\ x^1_t-\epsilon & \text{if } t=T+n_1-1\\ x^1_t & \text{otherwise}\end{array}\right. \qquad \tilde{x}^2_t=\left\{\begin{array}{ll}x^2_t-\epsilon & \text{if } t=T-1\\ x^2_t+\epsilon & \text{if } t=T+n_1-1\\ x^2_t & \text{otherwise}\end{array}\right.$$

which is feasible for $\epsilon > 0$ small, so that its benefit is smaller than the optimum. Since at time T we reach the state $\hat{\mathbb{X}}$ we deduce

$$b^{n_1-n_2}[U_1(x^1_{T+n_1-n_2-1}+\epsilon)-U_1(x^1_{T+n_1-n_2-1})] + [U_2(x^2_{T-1}-\epsilon)-U_2(x^2_{T-1})]$$
$$+b^{n_1}[U_2(\epsilon)-U_2(0) + U_1(\hat{x}^1_{n_1-1}-\epsilon)-U_1(\hat{x}^1_{n_1-1})] \leq 0$$

so that dividing as usual by ϵ and letting it to 0 we obtain

$$b^{n_1-n_2}U_1'(x^1_{T+n_1-n_2-1}) - U_2'(x^2_{T-1}) + b^{n_1}[U_2'(0) - U_1'(\hat{x}^1_{n_1-1})] \leq 0.$$

Since $x^2_t = 0$ for all $t \geq T$ it follows that $x^1_{T+n_1-n_2-1} \leq \hat{x}^1_{n_1-n_2-1(n_1)}$ and using the fact that $x^2_{T-1} > 0$ the last inequality yields

$$U_2'(0) > \sigma_1[\tfrac{1}{b^{n_2}}U_1'(\hat{x}^1_{n_1-n_2-1(n_1)}) - U_1'(\hat{x}^1_{n_1-1})]$$

contradicting $\hat{\mathbb{X}} \in \Delta^p$ (consider Theorem 2.3.4). This contradiction implies $x^2_{T-1} = 0$ and the area balance at stage $T-1$ gives $x^1_{T-1} = x^1_{T+n_1-1} = \hat{x}^1_{n_1-1}$ so that $\mathbb{X}_{T-1} \in \Delta^p$. We conclude inductively that Δ^p is reached within n_2 steps. ■

Proposition 2.5.6. *Suppose $n_2 > n_1$, $(n_1, n_2) = 1$ and $I^* = \{1\}$. For a given $\mathbb{X}_0 \in \Delta^g$ the optimal trajectory reaches Δ^p within n_2 steps if and only if $\mathbb{X}_0' \in \Delta^p$ where $X_0'^2 = 0$, $\bar{x}_0'^1 = 0$ and $x_t'^1 = x^1_t + \sum_{i=0}^{n_2-1} x^2_i \mathbb{1}_{\{i=t(n_1)\}}$ for all $t = 0, \ldots, n_1 - 1$.*

Proof. Obviously Δ^p may not be reached with a greedy policy in n_2 stages if $\mathbb{X}_0' \notin \Delta^p$. Conversely let us show that the trajectory $x^2_t = 0$ for $t \geq n_2$ and

$$x^1_t = \begin{cases} x^1_{t(n_1)} + \sum_{i=0}^{t-1} x^2_i \mathbb{1}_{\{i=t(n_1)\}} & t = n_1, \ldots, n_2 - 1 \\ x'^1_{t(n_1)} & t \geq n_2 \end{cases}$$

is optimal. We consider the Lagrangian in (2.12) and the following ℓ^1-multipliers

$$\begin{cases} \theta_t = b^t \sigma_1 U_1'(x^1_t) & t \geq n_2 \\ \theta_t = b^t \left[\sum_{j=1}^{\ell} b^{jn_1} U_1'(x^1_{t+jn_1})\right] + \theta_{t+\ell n_1} & t < n_2 \text{ where } t + \ell n_1 \in [n_2, n_1 + n_2) \\ \lambda^i_t = \theta_{t-n_i} - \theta_t - b^t U_i'(x^i_t) & t \geq n_i. \end{cases}$$

A cumbersome computation shows that $\nabla L = 0$ and that we have complementary slackness, so the proposed trajectory is optimal. ■

2.6 Conclusion

In this paper we discussed a model for the optimal management of a mixed forest composed by several species with different maturity ages, under the restriction that only mature trees can be harvested.

We established the existence and uniqueness of a *sustainable state* which is invariant under the optimal harvesting policy, and which was characterized as the unique solution of a finite dimensional optimization problem. We also discussed the existence of periodic optimal solutions and showed that if $(n_1, \ldots, n_{i^*}) = 1$ then the only periodic solution is the sustainable state.

In our main result we proved that any optimally managed forest must converge towards the set of greedy periodic cycles. In particular when $(n_1, \ldots, n_{i^*}) = 1$ all optimal trajectories converge to the sustainable state. The key for the asymptotic analysis was to identify a pseudo-Lyapunov function which increases every N steps where N is the least common multiple of the maturity ages $n_1, \ldots, n_k$ of the species considered.

As a by-product of the convergence analysis we obtained that in the long run any optimal trajectory becomes asymptotically greedy in the sense that the areas covered by over-mature trees converge to zero. We also described some particular cases in which the greedy regime is attained after a *finite* initial phase in which over-mature areas are required for transferring area between different age classes and species. However the finite time convergence to a greedy regime remains an open question in the general case. On the other hand, we showed that the convergence towards the set of greedy periodic cycles may in fact be only asymptotic and require an infinite time, even if we start from an initial state whose corresponding optimal solution is known to be greedy. More precisely, for a two species forest we characterized the set of initial conditions for which there is finite time convergence to a greedy periodic cycle, which turned out to be a null Lebesgue measure subset of Δ. This is a significant difference with respect to a one species forest which always converges in finite time to an optimal greedy periodic cycle.

Another open problem is that, although we established the convergence of $\mathbb{X}_t$ towards the set Δ^p, we did not prove the convergence to a particular GPC, except of course when $\Delta^p = \{\mathbb{X}^*\}$. A positive answer to this question and the characterization of the basins of attraction would be of interest. In the simple case of a two species forest with maturity ages $n_1 = 2$ and $n_2 = 1$ we could prove convergence to a particular GPC which depends on the initial condition, but the analysis is too specific and does not extend to higher dimensions so we did not present it.

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Chapter 3

Asymptotic convergence of optimal harvesting policies for a multiple species forest with a land market

This chapter corresponds to the article *The optimal harvesting problem with a land market: a characterization of the asymptotic convergence* [2], submitted.

Abstract

We study the asymptotic behavior of the optimal harvesting policies for a mixed forest with multiple species with a land market, i.e., any fraction of the land can be traded at any time stage. We prove the existence of *sustainable states* and we discuss the conditions under which any optimal trajectory converges in the long run towards one of these states or towards an optimal periodic cycle. We also discuss briefly the more general problem of a forest where conversion of the liberated land between the different species is costly.

Key words. forest management, discrete dynamic programming, infinite horizon, asymptotic convergence, Lyapunov stability

AMS subject classification. 93C55, 93D05, 93D20, 90B50

In abstract terms our model concerns the optimal management of a finite resource which can be allocated to different activities that, after a fixed delay, provide a benefit and liberate the resource for immediate reuse. At any time stage, any fraction of the resource can be traded. We study in particular, the optimal harvesting problem where the land surface is allocated among several forest species.

The exploitation of forests to get economic benefit dates from many centuries and the first ideas to make this benefit maximal appeared as early as the 15th century (for an historical account see [34, 37]). In 1849, Martin Faustmann correctly specified the problem of finding the economic value of an even-aged forest stand [13], a question solved by Ohlin in 1921 proving the existence of an optimal rotation period and characterizing the so-called *Faustmann age* [26]. The simplicity of Ohlin's result stems from the fact that he dealt with forests of identically aged trees. The generalization of the optimal rotation problem to a forest with many even-aged stands was already considered at Faustmann's time, but its complete resolution remains open even today. Nevertheless, Faustmann's ideas were extremely influential and inspired various harvesting rules which present a long run behavior that guarantees a sustainable and regular flow of timber. Today, the forest economics literature proposes many different ways of tackling the problem. Solutions range from simple heuristics, simulations, linear optimization models, optimal control problems, exhaustive search of solutions, etc. Most of these works assume a priory that the desirable long-run state of the forest's population is some even land allocation between tree stands, or include different types of even-flow constraints. Only a few works have dealt with the general problem without additional assumptions. Among them, it stands out the ground-breaking work of Mitra & Wan [22], which studies for the first time the problem of a multi-aged single species forest. They prove the existence of a state invariant under the optimal trajectory, the so-called *sustainable state*, but they also show that the popular belief about the long-run behavior of the forest state may not be true: in the general case there is no convergence to one homogeneous state. The articles by Salo & Tahvonen [31]-[33] improve these results using mathematical programming techniques. Also, Rapaport, Sraidi and Terreaux [30] consider a single species forest where harvest is forbidden before the maturity age and such that the tree's value remains constant after this age, showing that every optimal trajectory becomes periodic after a finite time, which means that the problem can be reduced to a finite dimensional one and easily solved.

In [1], Cominetti and Piazza extend the model of Rapaport et al. to a mixed forest with k-species having different maturity ages. They prove the existence and uniqueness of a sustainable state and provide sufficient conditions under which this state is a global attractor for the optimal trajectories. If these conditions are not satisfied the sustainable state is still a fixed point but optimal trajectories converge to the larger set of optimal periodic cycles.

In the present paper we extend the model in [1] to include the possibility of buying and selling land at any time step. The analysis becomes more difficult because the uniqueness of the sustainable state is lost and new optimal periodic cycles appear, as we show in §3.2. Nevertheless, we prove in §3.3 that the main result is still valid under very general conditions: optimal trajectories converge towards one of the sustainable states or to the set of optimal periodic cycles when the optimal trajectory becomes greedy (§ 3.3.2), when the land price is strictly concave (§

3.3.3) and when this price is linear (§ 3.3.4). In every case, we show that land trading converges to zero and when the land price is linear this convergence occurs in finite time. Finally, in §3.4 we briefly discuss the asymptotic behavior of a forest where land conversion between species is costly. This problem is a generalization of the main topic of this article and the same type of asymptotic behavior is expected. It is worth mentioning that this issue was already considered by Salo and Tahvonen in [33] for the case of a two species forest when one of the species is annual, showing that conversion costs may induce new optimal periodic cycles.

3.1 Model formulation

We consider a discrete time model similar for the optimal management of a forest composed by k species $I = \{1, \dots, k\}$ with maturity ages of $n_1, \dots, n_k$ years respectively. For each period $t \in I\!N$ we denote $x_t^i \geq 0$ the area of species $i \in I$ reaching maturity in year t, while $\bar{x}_t^i \geq 0$ represents the area with trees beyond maturity (older than n_i). We must decide how much area $u_t^i \geq 0$ to harvest and how much land $c_t \in I\!R$ to trade on the market, after which the available land is allocated to new seedlings.

Let S denote the total surface and $a_t = \sum_{i \in I} \left[\bar{x}_t^i + \sum_{j=0}^{n_i - 1} x_{t+j}^i\right]$ the area occupied at time t. Taking $c_t > 0$ if some land is bought and $c_t < 0$ if it is sold, the occupied land evolves simply as

$$a_{t+1} = a_t + c_t, \qquad 0 \leq a_t \leq S. \tag{3.1}$$

Assuming that only mature trees can be harvested we must have $u_t^i \leq \bar{x}_t^i + x_t^i$. The area of mature trees not harvested in one period will comprise the trees beyond maturity at the following step

$$\bar{x}_{t+1}^i = \bar{x}_t^i + x_t^i - u_t^i. \tag{3.2}$$

The area made available after the harvest $\sum_{i \in I} u_t^i + c_t$ is allocated to new seedlings that will reach maturity in years $t + n_i$ respectively which is expressed by the equation

$$\sum_{i \in I} x_{t+n_i}^i \;=\; c_t + \sum_{i \in I} u_t^i. \tag{3.3}$$

The benefit obtained from the harvest is $\sum_{t=0}^{\infty} \sum_{i \in I} b^t U_i(u_t^i) + b^t W(a_t, c_t)$ where $b \in (0,1)$ is a discount rate and $U_i : I\!R \to I\!R$ are smooth, increasing and strictly concave functions that represent the benefit rendered by each of the forest species. The cost of land transactions is modelled as

$$W(a, c) = -\int_a^{a+c} \rho(\xi) d\xi - \gamma |c|$$

where $\rho(\cdot) : [0, S] \to I\!R_+$ is a continuous, non-decreasing price function which takes into account that the scarcer the land is, the more expensive it becomes, while the term $-\gamma|c|$ incorporates transaction costs such as administrative expenses. Notice that $W(\cdot, c)$ is decreasing and concave. We also require $\rho(0) > \gamma$, which assures that $W(\cdot, c) > 0$ whenever $c < 0$, i.e., benefit is always obtained when land is sold.

We denote $\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i, \boldsymbol{a}, \boldsymbol{c}$ the sequences of states and controls. Since all areas are smaller than S and only $\boldsymbol{c}$ can take negative values, being bounded below by $-S$, it follows that these sequences belong to the ℓ^∞ ball B_S^∞ of radius S and centered at the origin. An alternative representation of the forest in terms of the age distribution at time t is provided by the *state* $\mathbb{X}_t = (X_t^1, \ldots, X_t^k)$ where $X_t^i = (x_{t+n_i-1}^i, x_{t+n_i-2}^i, \ldots, x_t^i, \bar{x}_t^i)$ describes the areas occupied in year t by trees of species i with ages $1, 2, \ldots, n_i$ and over n_i. The state evolution consists of an age-shift dynamics, except for the first and last components of each vector X_t^i which are controlled by the harvesting/sowing policy. Although we will not use these dynamics explicitly, the state $\mathbb{X}_t$ will be useful in describing the asymptotic behavior of the forest. Notice that we do not control $\mathbb{X}_0$ which corresponds to the initial state reflecting the age-class composition of the forest at time $t = 0$, so that the problem to be solved may be stated as

$$P(\mathbb{X}_0) \quad \begin{cases} \text{maximize} \quad \sum_{t=0}^{\infty} b^t \big[\sum_{i \in I} U_i(u_t^i) + W(a_t, c_t) \big] \\ \text{subject to (3.1)-(3.3)} \\ \boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i, \boldsymbol{a} \in \ell_+^\infty, \ \boldsymbol{c} \in \ell^\infty \text{ with } \mathbb{X}_0 \text{ given.} \end{cases}$$

We denote Δ the set of all initial states $\mathbb{X}_0 \in I\!R_+^{\sum_{i \in I}(n_i+1)}$ such that $\sum_{i \in I} [\bar{x}_0^i + \sum_{t=0}^{n_i-1} x_t^i] \leq S$. Clearly enough the constraints in $P(\mathbb{X}_0)$ imply that $\mathbb{X}_t \in \Delta$ for all $t \in I\!N$. We also denote Δ^0 the set of states with $\bar{x}_0^i = 0$ for all $i \in I$, and we observe that an initial state $\mathbb{X}_0 \in \Delta$ yields the same optimal value and harvesting policy as $\tilde{\mathbb{X}}_0 \in \Delta^0$ where $\tilde{X}_0^i = (x_{n_i-1}^i, x_{n_i-2}^i, \ldots, \bar{x}_0^i + x_0^i, 0)$.

Proposition 3.1.1. *For each $\mathbb{X}_0 \in \Delta$ the problem $P(\mathbb{X}_0)$ has an optimal solution.*

This proposition is a consequence of the Weierstrass theorem because the feasible set is non empty and $\sigma(\ell^\infty, \ell^1)$-compact while the objective function is $\sigma(\ell^\infty, \ell^1)$-upper semi continuous. The proof is a straightforward adaptation of [1] where a simpler problem with no land trading is treated. We observe that the harvests $\boldsymbol{u}^i$ are unique due to the strict concavity of U_i. Hence, we can assure the uniqueness of the $\boldsymbol{x}^i$ when restricted to so-called greedy trajectories where $\bar{\boldsymbol{x}}^i = 0$. Uniqueness of $\boldsymbol{a}$ and $\boldsymbol{c}$ follows as well from (3.1) and (3.3).

Before going on with the resolution of the problem, we express it in a different way. The objective function can be stated as

$$\begin{aligned} V &= \sum_t b^t \big[\sum_{i=1}^k U_i(u_t^i) - \int_0^{a_{t+1}} \rho(\xi) d\xi + \int_0^{a_t} \rho(\xi) d\xi - \gamma |c_t| \big] \\ &= \tfrac{1}{b} \int_0^{a_0} \rho(\xi) d\xi + \sum_t b^t \big[\sum_{i=1}^k U_i(u_t^i) - \tfrac{1-b}{b} \int_0^{a_t} \rho(\xi) d\xi - \gamma |c_t| \big] \end{aligned}$$

where the first term depends only on the initial conditions. Also, defining the variable $x_t^0 = S - a_t$ which represents the unused area, we get $c_t = x_t^0 - x_{t+1}^0$ and the area balance can be written as

$$\sum_{i \in I} x_{t+n_i}^i + x_{t+1}^0 = x_t^0 + \sum_{i \in I} u_t^i.$$

This suggests to consider the unused land as a new species X^0 with benefit function

$$U_0(x^0) = -\frac{1-b}{b}\int_0^{S-x^0} \rho(\xi)d\xi$$

and maturity age $n_0 = 1$. Notice that $U_0(\cdot)$ is smooth, increasing and concave as a function of x^0. The fact that U_0 is negative is of no importance to our proofs. With these definitions we may restate the problem as

$$P(\mathbb{X}_0)\begin{cases} \text{maximize} & \sum_{t=0}^{\infty} b^t \big[\sum_{i=0}^{k} U_i(u_t^i) - \gamma|c_t|\big] \\ \text{subject to} & \bar{x}_{t+1}^i = \bar{x}_t^i + x_t^i - u_t^i \quad i \in \{0, \cdots, k\} \\ & \sum_{i=0}^{k} x_{t+n_i}^i = \sum_{i=0}^{k} u_t^i \\ & x_{t+1}^0 = x_t^0 - c_t \\ & \boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i \in \ell_+^\infty, \boldsymbol{c} \in \ell^\infty \text{ with } \mathbb{X}_0 \text{ given.} \end{cases}$$

Notice that having $n_0 = 1$ and U_0 increasing assure that $\bar{\boldsymbol{x}}^0 = 0$ and $\boldsymbol{u}^0 = \boldsymbol{x}^0$ at the optimum.

3.2 Stationary optimal trajectories

In this section we characterize the initial states $\mathbb{X}_0$ that give rise to stationary or periodic optimal trajectories with no land trading. In the next section we discuss conditions under which every optimal trajectory converges to this particular set of states.

3.2.1 Sustainable states

We begin by introducing the notion of a sustainable state, which corresponds intuitively to a forest with an age distribution at which it is optimal to stay forever.

Definition 3.2.1. A state $\mathbb{X} \in \Delta$ is called *sustainable* if it is invariant under an optimal policy. The set of sustainable states is denoted Δ^*.

The existence of sustainable states is not completely obvious. Clearly any such state must be of the form $X^i = (x^i, \ldots, x^i, \bar{x}^i)$ with an invariant optimal harvesting policy: harvest x^i and sow exactly the same area in order to keep an invariant configuration. It is also clear that we must have $\bar{x}^i = 0$, that is to say $\mathbb{X} \in \Delta^0$, since otherwise a policy that harvests a little more at time $t = 0$ and x^i in all other periods would provide a greater benefit contradicting optimality. For the rest of this paper we denote $\sigma_i = b^{n_i}/(1-b^{n_i})$ and without loss of generality we assume that the species are ordered in such a way that $\sigma_1 U_1'(0) \geq \sigma_2 U_2'(0) \geq \cdots \geq \sigma_k U_k'(0)$.

In order to characterize the sustainable states, for each area $a \in [0,S]$ we denote $\mathbb{X}_a^* \in \Delta^0$ the homogeneous state $X_a^i = (x_a^{*i}, \ldots, x_a^{*i}, 0)$ where x_a^* is the solution to the strictly concave problem

$$(S_a) \quad \begin{cases} \text{maximize} & \sum_{i\in I} n_i \sigma_i U_i(x^i) \\ \text{subject to} & x^i \geq 0 \text{ and } \sum_{i\in I} n_i x^i = a. \end{cases}$$

We denote $I_a^* = \{i \in I : x_a^{*i} > 0\}$ the species present in $\mathbb{X}_a^*$ and we let r_a be the Lagrange multiplier associated to the area constraint $\sum_{i\in I} n_i x_a^i = a$, so that the optimal solution is characterized by $\sigma_i U_i'(x_a^{*i}) = r_a$ for $i \in I_a^*$ and $\sigma_j U_j'(0) \leq r_a$ for $j \notin I_a^*$. The ordering of the species and the strict concavity of U_i then imply that $I_a^* = \{1, \ldots, i_a^*\}$ for some index i_a^*.

We will prove that every sustainable state is of the form $\mathbb{X}_a^*$ for some area $a \in [0,S]$. However not every surface a supports a sustainable state. In order to characterize these surfaces we remark that when a increases all the optimal areas x_a^{*i} increase while the multiplier r_a decreases.

Definition 3.2.2. Let $\delta : [0,S] \to \mathbb{R}$ be the increasing function $\delta(a) = \rho(a) - r_a$ and denote $[\underline{a}, \overline{a}]$ the interval which comprises all the areas $a \in (0,S)$ such that $|\delta(a)| \leq \gamma$, together with 0 if $\delta(0) \geq -\gamma$ and S if $\delta(S) \leq \gamma$. An alternative expression for $\delta(a)$ is $\sigma_0 U_0'(S-a) - \sigma_1 U_1'(x_a^{*1})$.

Notice that when $\delta(0) \geq \gamma$ we have a degenerate interval $[\underline{a}, \overline{a}] = \{0\}$, while when $\delta(S) \leq -\gamma$ we get $[\underline{a}, \overline{a}] = \{S\}$. In all other cases $[\underline{a}, \overline{a}]$ is always non-empty and non-degenerate.

Proposition 3.2.3. $\Delta^* = \{\mathbb{X}_a^* : a \in [\underline{a}, \overline{a}]\}$.

Proof. Let us prove that $\mathbb{X}_a^*$ is sustainable for all $a \in [\underline{a}, \overline{a}]$. Take $a \in (0,S)$ with $|\delta(a)| \leq \gamma$ and consider the stationary trajectory issued from the initial condition $\mathbb{X}_0 = \mathbb{X}_a^*$. In order to prove its optimality for $P(\mathbb{X}_a^*)$ we introduce the Lagrangian

$$(3.4) \quad L = \sum_{t=0}^\infty \left\{ b^t \left[\sum_{i=0}^k U_i(u_t^i) - \gamma |c_t| \right] + \sum_{i=0}^k \mu_t^i u_t^i \right\} + \sum_{i=0}^k \left[\sum_{t=1}^\infty \bar{\lambda}_t^i \bar{x}_t^i + \sum_{t=n_i}^\infty \lambda_t^i x_t^i \right]$$
$$+ \sum_{t=0}^\infty \left\{ \sum_{i=0}^k \alpha_t^i (\bar{x}_t^i + x_t^i - u_t^i - \bar{x}_{t+1}^i) + \theta_t \sum_{i=0}^k (u_t^i - x_{t+n_i}^i) + \nu_t (x_t^0 - c_t - x_{t+1}^0) \right\}$$

together with the following set of ℓ^1-multipliers

$$(3.5) \quad \begin{cases} \mu_t^i = \lambda_t^0 = 0 \\ \theta_t = b^t r_a \\ \alpha_t^i = \theta_t + b^t U_i'(x_a^{*i}) & i \in \{0, \cdots, k\} \\ \bar{\lambda}_t^i = \alpha_t^i (1-b)/b & i \in \{0, \cdots, k\} \\ \lambda_t^i = \frac{\theta_t}{\sigma_i} - b^t U_i'(x_a^{*i}) & i \in I \\ \nu_t = b^t \delta(a). \end{cases}$$

A routine calculation shows that $0 \in \partial L$, and that the complementary slackness is satisfied with $\theta, \alpha^i, \bar{\lambda}^i, \lambda^i \in \ell_+^1$. Thus the proposed trajectory is a stationary point of the Lagrangian, hence an optimal solution of $P(\mathbb{X}_0)$ and therefore $\mathbb{X}_a^*$ is sustainable.

Similarly, to show that $\mathbb{X}_0^*$ is a sustainable state when $\delta(0) \geq -\gamma$, we take the multipliers

$$\begin{cases} \mu_t^i = \lambda_t^0 = 0 \\ \theta_t = b^t[\sigma_0 U_0'(S) + \gamma] \\ \alpha_t^i = \theta_t + b^t U_i'(x_0^{*i}) & i \in \{0, \cdots, k\} \\ \bar{\lambda}_t^i = \alpha_t^i(1-b)/b & i \in \{0, \cdots, k\} \\ \lambda_t^i = \frac{\theta_t}{\sigma_i} - b^t U_i'(0) & i \in I \\ \nu_t = -b^t\gamma. \end{cases}$$

The stationarity of the Lagrangian and the complementary slackness are straightforward, while the non-negativity of λ_t^i follows from $\lambda_t^i \geq \frac{b^t}{\sigma_i}[\delta(0) + \gamma] \geq 0$. Finally, to see that $\mathbb{X}_S^*$ is a sustainable state when $\delta(S) \leq \gamma$, consider again the multipliers of (3.5) except for the values of λ_t^0 and ν_t

$$\begin{cases} \lambda_t^0 = \frac{b^t}{\sigma_0}[\gamma - \delta(S)] \geq 0 \\ \nu_t = b^t\gamma. \end{cases}$$

Conversely, we show next that every sustainable state is of the form $\mathbb{X}_a^*$ for some $a \in [\underline{a}, \overline{a}]$. Let $\mathbb{X}$ be sustainable with $X^i = (x^i, \ldots, x^i, 0)$ and set $a = \sum_{i \in I} n_i x^i$. We claim that $x^i > 0$ implies $\sigma_i U_i'(x^i) \geq \sigma_j U_j'(x^j)$ for all $j \in I$. Indeed, let us perturb the optimal harvesting policy as follows: at time $t = 0$ we sow $x^i - \epsilon$ and $x^j + \epsilon$ instead of x^i and x^j, while in all subsequent periods we harvest all mature trees and sow the harvested areas with the same species they had. The benefit derived from this perturbed policy must be less than the one obtained with the optimal policy, which gives

$$\frac{b^{n_i}}{1-b^{n_i}}[U_i(x^i - \epsilon) - U_i(x^i)] + \frac{b^{n_j}}{1-b^{n_j}}[U_j(x^j + \epsilon) - U_j(x^j)] \leq 0.$$

Dividing by ϵ and letting $\epsilon \to 0$ we deduce $\sigma_i U_i'(x^i) \geq \sigma_j U_j'(x^j)$ as claimed. Using this fact and setting $I^* = \{i : x^i > 0\}$ it follows that $\sigma_i U_i'(x^i)$ is constant for $i \in I^*$ and larger than the value of this expression for $i \notin I^*$. This implies that the vector $(x^i)_{i \in I}$ is an optimal solution for (S_a) so that $x^i = x_a^{*i}$ and therefore $\mathbb{X} = \mathbb{X}_a^*$.

It is still left to see that $a \in [\underline{a}, \overline{a}]$, for which we must prove that $\delta(a) \geq -\gamma$ if $a < S$ as well as $\delta(a) \leq \gamma$ if $a > 0$. Suppose first that $a < S$ and consider the following perturbation to the optimal stationary trajectory issued from $\mathbb{X}_a^*$: at time $t = 0$ buy an ϵ of land and sow $x_a^{1*} + \epsilon$ instead of x_a^{1*}. After that we continue with a periodic policy, harvesting all mature trees and sowing the liberated areas with the same species they had. The benefit obtained with the alternative trajectory must be less or equal to the optimal one, which yields

$$\frac{b}{1-b}[U_0(x_a^{0*} - \epsilon) - U_0(x_a^{0*})] - \gamma|\epsilon| + \frac{b^{n_1}}{1-b^{n_1}}[U_1(x_a^{1*} + \epsilon) - U_1(x_a^{1*})] \leq 0.$$

Dividing by ϵ and letting $\epsilon \to 0$ we deduce $-\sigma_0 U_0'(x_a^{0*}) - \gamma + \sigma_1 U_1'(x_a^{1*}) \leq 0$, or equivalently $\delta(a) \geq -\gamma$. Similarly, if $a > 0$ we consider the selling of an ϵ of land, i.e., we take $\epsilon < 0$ and repeat the same reasoning to obtain $\sigma_0 U_0'(x_a^{0*}) - \gamma - \sigma_1 U_1'(x_a^{1*}) \leq 0$ which is equivalent to $\delta(a) \leq \gamma$. ■

3.2.2 Greedy Periodic Cycles

A trajectory is called *greedy* if all the mature trees are harvested at every stage. Such trajectories were introduced in [30] when studying a single species forest, in which case they yield periodic trajectories. This is no longer the case when multiple species are involved since the sowing policy is not determined. This distinction was already made in [1], introducing a special class of greedy trajectories called *greedy periodic cycles*, in which all harvested areas are sown with the same species they had before the harvest. In the current setting, for the artificial species X^0 that represents the unused land, the latter condition imposes that no land is traded so that $c_t = 0$ and $x_t^0 = x^0$ for all t.

Definition 3.2.4. A feasible trajectory $(\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i)_{i\in I}$ is called *greedy* if $\bar{x}_t^i = 0$. It will be called a *greedy periodic cycle (GPC)* if in addition $x_{t+n_i}^i = u_t^i$. We denote Δ^g the set of initial states $\mathbb{X}_0 \in \Delta^0$ for which there exists an optimal trajectory which is greedy, and Δ^p those having an optimal trajectory that is a *GPC*.

It is worth mentioning that since the optimal harvests are uniquely determined, the area balance implies that an optimal trajectory issued from an initial state $\mathbb{X}_0 \in \Delta^g$ must be unique, namely a greedy one, and therefore Δ^g is forward invariant through optimal policies: $\mathbb{X}_t \in \Delta^g \Rightarrow \mathbb{X}_{t+1} \in \Delta^g$.

Theorem 3.2.5. *Let $\mathbb{X}_0 \in \Delta^0$ such that $x^0 \in (0, S)$ and consider the periodic sequences $(x_t^i)_{t\in\mathbb{N}}$ built from $\mathbb{X}_0$ with $x_{t+n_i}^i = x_t^i$. Then $\mathbb{X}_0 \in \Delta^p$ iff for all $i, j \in I$ and $t \in \mathbb{N}$ we have* [1]

$$(3.6)\qquad \begin{array}{lll} (a) & U_i'(x_t^i) \geq b\, U_i'(x_{t+1}^i) & \\ (b) & \sigma_i U_i'(x_t^i) \geq b^{n_j}\left[\sigma_i U_i'(x_{t+n_j}^i) + U_j'(x_t^j)\right] & \forall\, x_t^i > 0 \\ (c) & \sigma_0 U_0'(x^0) - \sigma_1 U_1'(x_t^1) \leq \gamma & \forall\, x_t^1 > 0 \\ (d) & \sigma_1 U_1'(x_t^1) - \sigma_0 U_0'(x^0) \leq \gamma & \forall\, x_t^1 \end{array}$$

Proof. The arguments are similar to those used in [1, Theorem 3.4]. In particular we observe that conditions (a) and (b) are exactly the necessary and sufficient conditions that characterize Δ^p in [1], where the occupied surface is constant and given. The proof is based on [1, Lemma 3.5] which states that condition (b) implies

$$(3.7)\qquad x_t^i > 0 \quad \Rightarrow \quad \sigma_i U_i'(x_t^i) \geq \sigma_j U_j'(x_t^j) \quad for\ i, j \in I$$

[1] Notice that it suffices to check condition $a)$ for $t = 0, 1, \ldots, n_i - 1$. Similarly $c)$ and $d)$ must only be checked for $t = 0, 1, \ldots, n_1 - 1$, while for condition $b)$ it suffices to check it for $t = 0, 1, \ldots, n_{ij} - 1$ where n_{ij} is the least common multiple of the maturity ages n_i and n_j.

which in turn gives

$$x_t^i > 0 \Rightarrow x_t^j > 0 \text{ for all } j < i, \ i, j \in I \tag{3.8}$$

since otherwise (3.7) would yield the contradiction $\sigma_i U_i'(x_t^i) \geq \sigma_j U_j'(0) \geq \sigma_i U_i'(0) > \sigma_i U_i'(x_t^i)$. Combining (3.7) and (3.8) we get

$$x_t^i > 0 \Rightarrow \sigma_i U_i'(x_t^i) = \sigma_j U_j'(x_t^j) \text{ for all } j < i. \tag{3.9}$$

<u>SUFFICIENT CONDITION:</u> To establish the sufficiency take $\mathbb{X}_0$ satisfying (3.6) and consider the corresponding GPC. In order to prove its optimality if suffices to check that it is a stationary point for the Lagrangian (3.4) with the following set of multipliers

$$\begin{cases} \mu_t^i = \lambda_t^0 = 0 & \\ \theta_t = b^t \sigma_1 U_1'(x_t^1) & \\ \alpha_t^i = \theta_t + b^t U_i'(x_t^i) & i \in \{0, \cdots, k\} \\ \bar{\lambda}_t^i = \alpha_{t-1}^i - \alpha_t^i & i \in \{0, \cdots, k\} \\ \lambda_t^i = b^t \{\sigma_1 [\frac{1}{b^{n_i}} U_1'(x_{t-n_i}^1) - U_1'(x_t^1)] - U_i'(x_t^i)\}, & i \in I \\ \nu_t = b^t [\sigma_0 U_0'(x^0) - \sigma_1 U_1'(x_t^1)]. & \end{cases} \tag{3.10}$$

All these multipliers are of the form b^t multiplied by some bounded sequence so they belong to ℓ^1. The non-negativity of θ_t and α_t^i is evident and that of $\bar{\lambda}_t^i$ follows directly from (3.6)a. For $\lambda_t^i \geq 0$ we observe that this is assured by condition (3.6)b whenever $x_{t-n_i}^1 > 0$, while when $x_{t-n_i}^1 = 0$ using the monotonicity of U_i' and the fact that $\sigma_1 U_1'(0) \geq \sigma_i U_i'(0)$ we get

$$\frac{\lambda_t^i}{b^t} = \sigma_1 [\frac{1}{b^{n_i}} U_1'(0) - U_1'(x_t^1)] - U_i'(x_t^i) \geq \sigma_1 [\frac{1}{b^{n_i}} - 1] U_1'(0) - U_i'(x_t^i) \geq U_i'(0) - U_i'(x_t^i) \geq 0.$$

Therefore all the multipliers, except ν_t, belong to ℓ_+^1.

The complementary slackness is obvious except for the constraint $x_t^i \geq 0$ which follows from (3.9) because $x_t^i > 0$ implies $\sigma_i U_i'(x_t^i) = \sigma_1 U_1'(x_t^1) = \sigma_1 U_1'(x_{t-n_i}^1)$ and then $\lambda_t^i = 0$. Verification of stationarity is straightforward except for condition $0 \in \partial_{c_t} L$ which is equivalent to $\nu_t \in b^t[-\gamma, \gamma]$. This last condition is obviously assured by (c) and (d) when $x_t^1 > 0$. And of course, $\sigma_0 U_0'(x^0) - \sigma_1 U_1'(0) \leq \sigma_0 U_0'(x^0) - \sigma_1 U_1'(x_t^1) \leq \gamma$ which proves (c) for all t as long as $X^1 \neq 0$. But having $X^1 = 0$ is impossible since (3.8) would imply $X^i = 0$ for all $i \in I$ and $x^0 = S$. This completes the proof of optimality of the GPC for $P(\mathbb{X}_0)$.

<u>NECESSARY CONDITION:</u> To prove the necessity of condition (3.6) take $\mathbb{X}_0 \in \Delta^p$ so that the periodic sequences $(x_t^i)_{t \in I\!N}$ are optimal for $P(\mathbb{X}_0)$. The key idea is to compare the optimal benefit with those of a set of carefully chosen alternative trajectories.

We skip the proof of the necessity of (a) and (b) as it is identical to that of [1, Theorem 3.4].

To prove (3.6)(d) consider the following perturbation to the optimal trajectory: buy a fraction ϵ of land at time t and allocate it to the first species. The difference of benefit is

$$b^t \tfrac{b^{n_1}}{1-b^{n_1}}[U_1(x_t^1+\epsilon) - U_1(x_t^1)] + b^t \tfrac{b}{1-b}[U_0(x^0-\epsilon) - U_0(x^0)] - b^t\gamma\epsilon \leq 0$$

Thus, dividing by $b^t\epsilon$ and making $\epsilon \to 0$ we obtain $\sigma_1 U_1'(x_t^1) - \sigma_0 U_0'(x^0) \leq \gamma$.

A similar argument yields (3.6)(c): if $x_t^1 > 0$, we sell an ϵ of the land allocated to the first species and we repeat the same procedure to get $\sigma_0 U_0'(x^0) - \sigma_1 U_1'(x_t^1) \leq \gamma$. ■

The following corollary is a direct consequence of (3.9).

Corollary 3.2.6. *Let* $I(\mathbb{X}_0) = \{i : X^i \neq 0\}$. *Then for all* $\mathbb{X}_0 \in \Delta^p$ *we have* $I(\mathbb{X}_0) = \{1, \cdots, i_0\}$ *for some* i_0.

It may be surprising that (c) and (d) only involve species 0 and 1. In fact, there is an equivalent symmetric characterization,

Proposition 3.2.7. *Conditions* (3.6) *(c) and (d) of the theorem above can be substituted by*

$$\text{(3.11)} \qquad \begin{array}{ll} (c') & \sigma_0 U_0'(x^0) - \sigma_i U_i'(x_t^i) \leq \gamma \quad \forall\, x_t^i > 0,\ i \in I \\ (d') & \sigma_i U_i'(x_t^i) - \sigma_0 U_0'(x^0) \leq \gamma \quad \forall\, x_t^i,\ i \in I \end{array}$$

Proof. From (3.9) it is clear that (3.6)(c) and (3.11)(c') are equivalent. For (d) we observe that either $x_t^1 > 0$ and $\sigma_1 U_1'(x_t^1) \geq \sigma_i U_i'(x_t^i)$ or $x_t^1 = x_t^i = 0$ and $\sigma_1 U_1'(0) \geq \sigma_i U_i'(0)$. In both cases, (3.6)$(d)$ implies (3.11)(d') while the implication in the other sense is evident. ■

Let us consider next the limiting cases $x^0 = 0$ and $x^0 = S$. When $x^0 = S$ all the state variables are zero and the corresponding greedy periodic cycle is the stationary trajectory starting from $\mathbb{X}_0^* = 0$, which is optimal if and only if $\delta(0) \geq -\gamma$. For the case $x^0 = 0$ things are more involved. In fact, it is easy to see that conditions (3.6)(a), (b) and (c) are still necessary, but it turns out that they are not sufficient. Even more, we present a set of more restrictive necessary conditions which are not yet sufficient and a set of sufficient conditions. Both sets comprise conditions (3.6)(a) and (b) together with an ad hoc modification of (3.6)(c), while (d) is no longer considered as it is related to the possibility of buying land which is obviously not possible when $x^0 = 0$.

Theorem 3.2.8. *Consider a GPC issued from an initial state* $\mathbb{X}_0 \in \Delta^0$ *with* $x^0 = 0$. *The following conditions are sufficient to guarantee that* $\mathbb{X}_0 \in \Delta^p$: *for all* $i, j \in I$ *and* $t \in \mathbb{N}$ *we have*

$$\text{(3.12)} \qquad \begin{array}{lll} (a) & U_i'(x_t^i) \geq b\, U_i'(x_{t+1}^i) & \\ (b) & \sigma_i U_i'(x_t^i) \geq b^{n_j}\left[\sigma_i U_i'(x_{t+n_j}^i) + U_j'(x_t^j)\right] & \forall\, x_t^i > 0 \\ (c) & \sigma_1\left[U_1'(x_t^1) - bU_1'(x_{t+1}^1)\right] - bU_0'(x^0) + (1-b)\gamma \geq 0 & \forall\, x_t^1 > 0. \end{array}$$

A set of necessary conditions comprises (a), (b) *and* [2]

(3.13)

$$\sigma_i U_i'(x_t^i) - b^k \sigma_j U_j'(x_{t+k}^j) - (1-b^k)\sigma_0 U_0'(x^0) + (1+b^k)\gamma \geq 0 \quad \forall\, x_t^i > 0, i, j \in I \text{ and } k \in \mathbb{N}.$$

Proof. To prove the sufficiency of condition (3.12) it is enough to consider again the multipliers of (3.10) except for the values of λ_t^0 and ν_t

$$\begin{cases} \lambda_t^0 = b^t\{\sigma_1\big[U_1'(x_{t-1}^1) - bU_1'(x_t^1)\big] - bU_0'(0) + (1-b)\gamma\} \\ \nu_t = b^t\gamma. \end{cases}$$

The concavity of U_1 implies that if (3.12)(c) holds at time t then it also holds at $t+1$ even if $x_{t+1} = 0$. As (3.8) forces $X^1 \neq 0$, we deduce that (3.12)(c) holds for all t and the non-negativity of λ_t^0 follows directly.

On the other hand, the necessity of (3.13) is easily seen by taking a perturbed trajectory that consists in selling ϵ of the land assigned to species i at time t and buying it back at time $t+k$ to assign it to species j, continuing afterwards with a GPC. The difference of benefit is

$$\begin{aligned} &b^t \tfrac{b^{n_i}}{1-b^{n_i}}[U_i(x_t^i-\epsilon) - U_i(x_t^i)] + b^{t+k}\tfrac{b^{n_j}}{1-b^{n_j}}[U_j(x_{t+k}^j+\epsilon) - U_j(x_{t+k}^j)] \\ &+b^t(b+\cdots+b^k)[U_0(\epsilon) - U_0(0)] - b^t\gamma\epsilon - b^{t+k}\gamma\epsilon \leq 0 \end{aligned}$$

Thus, dividing by $b^t\epsilon$ and letting $\epsilon \to 0$ we obtain

$$-\sigma_i U_i'(x_t^i) + b^k\sigma_j U_j'(x_{t+k}^j) + b\tfrac{1-b^k}{1-b}U_0'(0) - (1+b^k)\gamma \leq 0$$

which is exactly (3.13). ∎

Remark 3.2.9. Notice that *(3.13)* can be easily deduced from *(3.11)*(c') and (d') since

$$\sigma_i U_i'(x_t^i) - \sigma_0 U_0'(x^0) + \gamma \geq 0 \geq b^k\big[\sigma_j U_j'(x_{t+k}^j) - \sigma_0 U_0'(x^0) - \gamma\big] \quad \forall x_t^i{>}0 \;\Rightarrow\; (3.13).$$

This is the reason why it is not relevant when $x^0 \in (0, S)$.

3.2.3 Relations between GPCs and sustainable states

Clearly, every stationary trajectory is a GPC so $\Delta^* \subseteq \Delta^p$. In the following we examine more closely the connections between greedy periodic cycles and sustainable states, pointing out a situation in which they coincide.

Proposition 3.2.10. *Let* $\mathbb{X} \in \Delta^p$*. Then* $a = S - x^0 \in [\underline{a}, \overline{a}]$.

[2] Notice that (3.6)(c) is retrieved by letting $k \to \infty$ in (3.13).

Proof. By Definition 3.2.2 it suffices to check that $\delta(a) \le \gamma$ if $a > 0$ and $\delta(a) \ge -\gamma$ if $a < S$.

Suppose $a > 0$. If there is any $x_t^1 \ge x_a^{*1}$ then (3.6)(c) implies $\gamma \ge \sigma_0 U_0'(x^0) - \sigma_1 U_1'(x_t^1) \ge \sigma_0 U_0'(x^0) - \sigma_1 U_1'(x_a^{*1}) = \delta(a)$. Suppose by contradiction that $x_t^1 < x_a^{*1}$ for all t. Due to (3.9) and the characterization of the sustainable state we know that for all $x_t^i > 0$

$$\sigma_i U_i'(x_t^i) = \sigma_1 U_1'(x_t^1) > \sigma_1 U_1'(x_a^{*1}) \ge \sigma_i U_i'(x_a^{*i}) \Rightarrow x_t^i < x_a^{*i},$$

and thus the area balance cannot be satisfied, which is obviously absurd.

Suppose next $a < S$. If there is any $x_t^1 \le x_a^{*1}$, then $-\delta(a) \le \gamma$ follows from (3.6)(d). Suppose that $x_t^1 > x_a^{*1}$ for all t, proceeding as before we conclude that for all $i \in I_a^*$ we have $\sigma_i U_i'(x_a^{*i}) = \sigma_1 U_1'(x_a^{*1}) > \sigma_1 U_1'(x_t^1) \ge \sigma_i U_i'(x_t^i)$, where the last inequality follows from (3.7). This gives $x_a^{*i} < x_t^i$ for all t and $i \in I_a^*$, and so the area balance cannot be satisfied and a contradiction arises. ■

As a direct consequence of this result and the commentary after Definition 3.2.2, it follows that when $\delta(0) \ge \gamma$ we have $\Delta^p = \Delta^* = \{\mathbb{X}_0^*\}$, while if $\delta(S) \le -\gamma$ then $\Delta^p \subseteq \{\mathbb{X} \in \Delta^0 : x^0 = 0\}$. The following result gives a sufficient condition to assure the equality. For the rest of the paper m.c.d. stands for *maximum common divisor*. From now on, we denote $\Delta^p(a) = \Delta^p \cap \{\mathbb{X} \in \Delta^0 : x^0 = S - a\}$.

Proposition 3.2.11. *If m.c.d.*$\{n_i : i \in I_a\} = 1$ *then* $\Delta^p(a) = \{\mathbb{X}_a^*\}$. *Thus, if m.c.d.*$\{n_i : i \in I_{\underline{a}}\} = 1$ *then* $\Delta^p = \Delta^*$.

Proof. Using [1, Theorem 3.8] we see that for a given surface a the first condition yields that $\mathbb{X}_a^*$ is the only state in Δ^p whose covered surface is equal to a. The second hypothesis implies m.c.d.$\{n_i : i \in I(\mathbb{X}_{\underline{a}}^*)\} = 1$ for all $a \in [\underline{a}, \overline{a}]$, which readily implies the lemma. ■

3.3 Convergence of optimal trajectories

We turn next to the study of the long run behavior of the optimal harvesting policies. The previous section described some special states from which the optimal trajectory is either invariant or periodic. We conjecture that such behavior is typical in the sense that an optimally managed forest converges either to a sustainable state or to the set of optimal GPCs. To prove such a *global attractor property* we rely on a suitable Lyapunov function to analyze the asymptotic behavior of optimal trajectories that become greedy after a finite time. We then present some sufficient conditions under which this is effectively the case. More importantly, we prove that when U_0 is strictly concave or linear this result holds not only for greedy trajectories but for every optimal harvesting policy. We also show that land trading converges to zero and that in the case of U_0 linear this convergence occurs after finitely many steps. In the very particular case of linear land price and $\gamma = 0$ we solve completely the problem determining explicitly the optimal trajectory.

3.3.1 Lyapunov function

We remark that the Lyapunov technique we use is somewhat non-standard since the energy does not increase at every stage but every N time steps, where N is the least common multiple of all the maturity ages n_i. We could recover a standard Lyapunov function by considering the sum or the maximum over N consecutive periods, however this would make the arguments unnecessarily obscure.

Let us define the function $\Phi : \Delta^0 \to I\!R$ given by

$$\Phi(\mathbb{X}_0) = G(\mathbb{X}_0) - \sum_{i\in I} \sum_{t=0}^{n_i-2} \tfrac{b^t - b^{n_i-1}}{1-b^{n_i}} U_i(x_t^i) \tag{3.14}$$

where $G(\mathbb{X}_0)$ is the optimal benefit obtained from state $\mathbb{X}_0$ by using a greedy policy

$$\begin{aligned} G(\mathbb{X}_0) \quad = \quad & \max \quad \textstyle\sum_{t=0}^{\infty} b^t \big[\sum_{i=0}^{k} U_i(x_t^i) - \gamma |c_t| \big] \\ & \text{s.t. } \boldsymbol{x}^i \in \ell_+^\infty \\ & \textstyle\sum_{i=0}^{k} x_{t+n_i}^i = \sum_{i=0}^{k} x_t^i \\ & x_{t+1}^0 = x_t^0 - c_t \end{aligned}$$

The clue for the subsequent asymptotic analysis is the following property which shows that Φ is a Lyapunov function.

Theorem 3.3.1. *Let N be the least common multiple of $\{n_i, i \in I\}$. If $\mathbb{X}_0 \in \Delta^g$ then*

$$\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0) + \gamma \textstyle\sum_{j=0}^{N-1} |c_j|$$

with strict inequality unless $\mathbb{X}_0 \in \Delta^p$. Hence Φ is a Lyapunov function modulo N.

Proof. To simplify the notation we set $U_t^i = U_i(x_t^i)$, $G_t = G(\mathbb{X}_t)$, and we denote

$$P_t = \textstyle\sum_{i=0}^{k} \frac{1}{1-b^{n_i}} \sum_{j=t}^{t+n_i-1} b^{j-t} U_j^i.$$

the benefit of a GPC started from state $\mathbb{X}_t$. Since G_t is the optimal benefit we have $G_t \geq P_t$ which can be written as $(1 - b^N)G_t \geq (1 - b^N)P_t$ and then

$$G_t \geq b^N G_t + \textstyle\sum_{i=0}^{k} \frac{1-b^N}{1-b^{n_i}} \sum_{j=t}^{t+n_i-1} b^{j-t} U_j^i.$$

Now, Bellman's principle of dynamic programming gives

$$\begin{aligned} G_0 &= \textstyle\sum_{j=0}^{t-1} b^j \big[\sum_{i=0}^k U_j^i - \gamma|c_j|\big] + b^t G_t \\ G_t &= \textstyle\sum_{j=t}^{N-1} b^{j-t} \big[\sum_{i=0}^k U_j^i - \gamma|c_j|\big] + b^{N-t} G_N \end{aligned}$$

which plugged into the previous inequality yields

$$\begin{aligned} b^{N-t}G_N + \sum_{j=t}^{N-1} b^{j-t}\Big[\sum_{i=0}^{k} U_j^i - \gamma|c_j|\Big] \;&\geq\; b^{N-t}G_0 - \sum_{j=0}^{t-1} b^{N-t+j}\Big[\sum_{i=0}^{k} U_j^i - \gamma|c_j|\Big] \\ &\quad + \sum_{i=0}^{k} \tfrac{1-b^N}{1-b^{n_i}} \sum_{j=t}^{t+n_i-1} b^{j-t} U_j^i. \end{aligned}$$

Adding up these equations for $t = 0, 1, \ldots, N-1$ we get

$$\begin{aligned} &\tfrac{b(1-b^N)}{1-b} G_N + \sum_{t=0}^{N-1}\sum_{j=t}^{N-1} b^{j-t}\Big[\sum_{i=0}^{k} U_j^i - \gamma|c_j|\Big] \geq \\ &\tfrac{b(1-b^N)}{1-b} G_0 - \sum_{t=0}^{N-1}\sum_{j=0}^{t-1} b^{N-t+j}\Big[\sum_{i=0}^{k} U_j^i - \gamma|c_j|\Big] + \sum_{t=0}^{N-1}\sum_{i=0}^{k} \tfrac{1-b^N}{1-b^{n_i}} \sum_{j=t}^{t+n_i-1} b^{j-t} U_j^i \end{aligned} \tag{3.15}$$

and using Fubini's rule to exchange all these multiple sums we deduce

$$\begin{aligned} \tfrac{b(1-b^N)}{1-b} G_N + \sum_{j=0}^{N-1} \tfrac{1-b^{j+1}}{1-b}\left[\sum_{i=0}^{k} U_j^i - \gamma|c_j|\right] \;&\geq\; \tfrac{b(1-b^N)}{1-b} G_0 - \sum_{j=0}^{N-1} \tfrac{b^{j+1}-b^N}{1-b}\left[\sum_{i=0}^{k} U_j^i - \gamma|c_j|\right] \\ &+ \sum_{i=0}^{k} \tfrac{1-b^N}{1-b^{n_i}} \left[\sum_{j=0}^{n_i-2} \tfrac{1-b^{j+1}}{1-b} U_j^i + \sum_{j=n_i-1}^{N-1} \tfrac{1-b^{n_i}}{1-b} U_j^i + \sum_{j=N}^{N+n_i-2} \tfrac{b^{j-N+1}-b^{n_i}}{1-b} U_j^i\right] \end{aligned}$$

The two sums on the first line may be combined and factored by the term $\frac{1-b^N}{1-b}$ which may then be dropped throughout in order to get

$$\begin{aligned} b\,G_N \;&\geq\; b\,G_0 - \sum_{j=0}^{N-1}\sum_{i=0}^{k} U_j^i + \sum_{j=0}^{N-1} \gamma|c_j| \\ &+ \sum_{i=0}^{k} \left[\sum_{j=0}^{n_i-2} \tfrac{1-b^{j+1}}{1-b^{n_i}} U_j^i + \sum_{j=n_i-1}^{N-1} \tfrac{1-b^{n_i}}{1-b^{n_i}} U_j^i + \sum_{j=N}^{N+n_i-2} \tfrac{b^{j-N+1}-b^{n_i}}{1-b^{n_i}} U_j^i\right] \end{aligned}$$

We may now change the order of summation of the first sum and cancel out the terms to deduce

$$b\,G_N \geq b\,G_0 + \sum_{j=0}^{N-1} \gamma|c_j| + \sum_{i\in I} \left[\sum_{j=0}^{n_i-2} \tfrac{b^{n_i}-b^{j+1}}{1-b^{n_i}} U_j^i + \sum_{j=N}^{N+n_i-2} \tfrac{b^{j-N+1}-b^{n_i}}{1-b^{n_i}} U_j^i\right]$$

and dividing by b, rearranging terms and taking into consideration that $n_0 = 1$ we have

$$\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0) + \gamma \textstyle\sum_{j=0}^{N-1} |c_j|. \tag{3.16}$$

When $\mathbb{X}_0 \notin \Delta^p$ we have $G_0 > P_0$ and therefore the inequality (3.15) as well as its consequence (3.16) become strict. ■

Corollary 3.3.2. *Along any optimal trajectory* $\lim_{t\to\infty} c_t = 0$ *and* $\lim_{t\to\infty} x_t^0$ *is well defined.*

Proof. First observe that Φ is bounded from above by some constant M. Then, proceeding by induction in (3.16) we obtain

$$M \geq \Phi(\mathbb{X}_{kN}) \geq \Phi(\mathbb{X}_0) + \gamma \sum_{t=0}^{kN-1} |c_t|$$

from which it follows that $\sum_{j=0}^{\infty} |c_j| < \infty$ and therefore $c_t \to 0$. Moreover $\{x_t^0\}$ is a Cauchy sequence since $|x_t^0 - x_{t+k}^0| \leq \sum_{j=t}^{t+k-1} |c_t|$, so the unused area x_t^0 has a well defined limit. ■

3.3.2 Asymptotic convergence for greedy trajectories

Since Δ^g is forward invariant under an optimal greedy strategy in the sense that $\mathbb{X}_0 \in \Delta^g$ implies $\mathbb{X}_t \in \Delta^g$ for all $t \geq 0$, it follows that $\Phi(\mathbb{X}_{t+N}) \geq \Phi(\mathbb{X}_t)$ showing that an optimal sequence of states $\mathbb{X}_t$ becomes an "N-step" uphill strategy for the Lyapunov function Φ as soon as it enters the set Δ^g. This property is used to prove that we have convergence to Δ^p. The proof is valid only when the optimal trajectory becomes greedy after finitely many steps. At the end of this subsection, we present some conditions under which this is assured.

Theorem 3.3.3. *Let* $\mathbb{X}_0 \in \Delta$ *be such that the optimal trajectory satisfies* $\mathbb{X}_t \in \Delta^g$ *for some* $t \in I\!N$. *Then the optimal trajectory converges to a GPC in the sense that*

$$\lim_{t\to\infty} \text{dist}(\mathbb{X}_t, \Delta^p) = 0. \tag{3.17}$$

Proof. We include the proof of this theorem, even though it was already presented in [1], because it is useful to understand the importance of the optimal trajectory's uniqueness, which forces to restrict the study to special situations.

It suffices to show that every accumulation point of $\mathbb{X}_t$ belongs to Δ^p. Suppose by contradiction that we have a sequence $t_j \to \infty$ with $\mathbb{X}_{t_j} \to \mathbb{X}^\infty \notin \Delta^p$, and assume with no loss of generality that $\mathbb{X}_{t_j} \in \Delta^g$ for all j. One of the sets $\{i + qN : q \in I\!N\}$ for $i = 1, \ldots, N$ contains infinitely many t_j's, so that passing to a subsequence we may further assume that $t_j = i + q_j N$ for a fixed i and $q_j \to \infty$.

The set-valued map which assigns to $\mathbb{X}_0 \in \Delta$ the solution set $S(\mathbb{X}_0)$ of $P(\mathbb{X}_0)$ is upper-semi-continuous with respect to the $\sigma(\ell^\infty, \ell^1)$ topology on ℓ^∞. This property combined with the fact that a weak* limit of a greedy trajectory is still greedy, implies that Δ^g is closed so that $\mathbb{X}^\infty \in \Delta^g$. On the other hand, after Definition 3.2.4 we observed that for $\mathbb{X}_0 \in \Delta^g$ *the optimal solution is unique so that the map* $\mathbb{X}_0 \mapsto S(\mathbb{X}_0)$ *is in fact strong-to-weak* continuous from* Δ^g *to* $(\ell^\infty)^{3k}$. It follows that the map which assigns to $\mathbb{X}_0 \in \Delta^g$ the state $\mathbb{X}_N \in \Delta^g$ reached at time N is continuous, and then the same holds for the function $\mathbb{X}_0 \in \Delta^g \mapsto \Phi(\mathbb{X}_N) \in I\!R$.

Now since $\mathbb{X}^\infty \not\in \Delta^p$, Theorem 3.3.1 gives $\Phi(\mathbb{X}^\infty_N) > \Phi(\mathbb{X}^\infty)$ and by continuity we may find $\epsilon > 0$ and a neighborhood $\mathcal{V}$ of $\mathbb{X}^\infty$ such that $\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0) + \epsilon$ for all $\mathbb{X}_0 \in \Delta^g \cap \mathcal{V}$. Since $\mathbb{X}_{t_j} \to \mathbb{X}^\infty$ we have $\mathbb{X}_{t_j} \in \Delta^g \cap \mathcal{V}$ for all j large, and then $\Phi(\mathbb{X}_{t_{j+1}}) \geq \Phi(\mathbb{X}_{t_j}) + \epsilon$. This implies $\Phi(\mathbb{X}_{t_j}) \to \infty$ which is impossible since Φ is bounded on Δ^g. This contradiction completes the proof. ■

Thanks to Corollary 3.3.2 we know that given $\mathbb{X}_0 \in \Delta$ the free land x_t^0 converges along the optimal trajectory. This allows us to state a sharpened version of the theorem above.

Corollary 3.3.4. *Let* $\mathbb{X}_0 \in \Delta$ *be such that the optimal trajectory satisfies* $\mathbb{X}_t \in \Delta^g$ *for some* $t \in I\!N$. *Let* $a_\infty = S - x^0_\infty$ *where* $x^0_\infty = \lim_{t\to\infty} x^0_t$. *Then*

$$\lim_{t\to\infty} \operatorname{dist}\big(\mathbb{X}_t, \Delta^p(a_\infty)\big) = 0.$$

If in addition m.c.d.$\{n_i : i \in I^*_{a_\infty}\} = 1$, *then the forest converges to the sustainable state*

$$\lim_{t\to\infty} \mathbb{X}_t = \mathbb{X}^*_{a_\infty}. \tag{3.18}$$

Proof. Let $\boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i, \boldsymbol{c}$ be an optimal trajectory for $P(\mathbb{X}_0)$. Given any $\epsilon > 0$ we may find T such that $\sum_{t=T}^\infty c_t \leq \frac{\epsilon}{2}$ and then

$$\operatorname{dist}\big(\mathbb{X}_t, \cup_{|a-a_\infty|>\epsilon} \Delta^p(a)\big) > \tfrac{\epsilon}{2} \quad \forall t \geq T.$$

Hence,

$$\operatorname{dist}(\mathbb{X}_t, \Delta^p) \to 0 \iff \operatorname{dist}(\mathbb{X}_t, \cup_{|a-a_\infty|\leq\epsilon} \Delta^p(a)) \to 0$$

and letting $\epsilon \to 0$ we get $\operatorname{dist}(\mathbb{X}_t, \Delta^p(a_\infty)) \to 0$. This result together with Proposition 3.2.11 readily imply (3.18). ■

The previous results are valid when the optimal trajectory is greedy or becomes greedy after finitely many steps. In [1], it is proved that this is the case if there is an annual forest species or when conditions $U_i'(1) \geq bU_i'(0)$ for every $i \in I$ are satisfied. These two sufficient conditions are still valid for our model and we also have the following

Proposition 3.3.5. *If* $\rho(\cdot) \geq \frac{1+b}{1-b}\gamma$ *then* $\bar{x}^i_t = 0$ *for all* $t \geq 2n_i - 1$ *and hence* $\mathbb{X}_t \in \Delta^g$ *for all* $t \geq 2\bar{n} - 1$ *where* $\bar{n} = \max_{i\in I} n_i$.

Proof. First note that in each interval of length n_i such as $p+1,\dots,p+n_i$ there is at least one $\bar{x}_t^i = 0$. Indeed, if this was not the case then at time p we could harvest a small additional area $\epsilon > 0$ and resow it immediately as species i, modifying the trajectory as $\bar{x}_t^i - \epsilon$ for $t = p+1,\dots,p+n_i$ and $x_{p+n_i}^i + \epsilon$, after which we rejoin the original optimal strategy. This modified trajectory would increase the benefit by an amount $b^p[U_i(u_p^i + \epsilon) - U_i(u_p^i)] > 0$ contradicting optimality. We remark that this holds for every function ρ.

Next observe that for $t \geq n_i$ we have $\bar{x}_t^i = 0 \Rightarrow \bar{x}_{t+1}^i = 0$. To see this we proceed again by contradiction: if $\bar{x}_t^i = 0 < \bar{x}_{t+1}^i$ then $u_t^i < \bar{x}_t^i + x_t^i = x_t^i$ which means that at stage t we do not harvest all the available trees of species i. If we backtrack to stage $t - n_i$ when x_t^i was sown, we could have sown $x_t^i - \epsilon$ and $x_{t-n_i+1}^0 + \epsilon$ instead of x_t^i and $x_{t-n_i+1}^0$, after which this ϵ is returned to species i so that at stage $t+1$ the trees reaching maturity $x_{t+1}^i + \epsilon$ compensate the loss of over-mature trees $\bar{x}_{t+1}^i - \epsilon$. This trajectory allows to harvest the same areas as in the original strategy, except at stage $t - n_i + 1$ the difference of benefit being

$$\begin{aligned} V_\epsilon - V &= b^{t-n_i}\gamma(-|x_{t+n_i+1}^0 + \epsilon - x_{t+n_i}^0| + |x_{t+n_i+1}^0 - x_{t+n_i}^0|) \\ &\quad + b^{t-n_i+1}[U_0(x_{t-n_i+1}^0 + \epsilon) - U_0(x_{t-n_i}^0)] \\ &\quad + b^{t-n_i+1}\gamma[-|x_{t+n_i+2}^0 - x_{t+n_i-1}^0 - \epsilon| + |x_{t+n_i+2}^0 - x_{t+n_i+1}^0|\,] \\ &\geq b^{t-n_i+1}\big[U_0(x_{t-n_i+1}^0 + \epsilon) - U_0(x_{t-n_i}^0) - (1 + \tfrac{1}{b})\epsilon\gamma\big] \end{aligned}$$

Dividing by $\big(b^{t-n_i+1}\frac{\epsilon}{\sigma_0}\big)$ and letting $\epsilon \to 0$ we get $\sigma_0 U_0'(x_{t-n_i+1}^0) - \frac{1+b}{1-b}\gamma = \rho(1 - x_{t-n_i+1}^0) - \frac{1+b}{1-b}\gamma > 0$ which contradicts optimality.

Combining the previous properties we may conclude: in the interval $n_i,\dots,2n_i - 1$ there exists at least one t such that $\bar{x}_t^i = 0$, condition that must hold thereafter. ■

3.3.3 Asymptotic convergence when U_0 is strictly concave

The uniqueness of the optimal trajectory plays a fundamental role in the proof of Theorem 3.3.3. If U_0 is strictly concave, then the area balance allows having up to one utility function merely concave non-decreasing without loosing uniqueness. We will use this observation to extend the result from greedy to general optimal trajectories.

Theorem 3.3.6. *Let* $\mathbb{X}_0 \in \Delta$. *Let* $a_\infty = S - x_\infty^0$ *where* $x_\infty^0 = \lim_{t\to\infty} x_t^0$. *Then*

$$\lim_{t\to\infty} \operatorname{dist}\big(\mathbb{X}_t, \Delta^p(a_\infty)\big) = 0.$$

If in addition m.c.d.$\{n_i : i \in I_{a_\infty}^*\} = 1$, *then the forest converges to the sustainable state*

$$\lim_{t\to\infty} \mathbb{X}_t = \mathbb{X}_{a_\infty}^*.$$

Proof. Clearly the proof is identical to that of Corollary 3.3.4 as long as we can state

$$\lim_{t\to\infty} \text{dist}(\mathbb{X}_t, \Delta^p) = 0. \tag{3.19}$$

To see (3.19) we adapt the proof of [1, Theorem 4.8]: every optimal trajectory has an equivalent greedy trajectory in an augmented problem where a dummy variable corresponding to *"bare land"* is added. This new variable must not be confused with the unused land x^0. The bare land is land that belongs to the forest owner but is neither allocated to any forest species nor sold, and hence its benefit function is null. This variable gives an extra degree of freedom allowing us to eliminate every overmature tree after some finite time, so that the augmented equivalent trajectory becomes greedy and we may apply the convergence result for this particular case, from which the conclusion in the general case follows. We obtain as a by product of the proof that the optimal trajectory is *asymptotically greedy*, i.e., $\lim_{t\to\infty} \bar{x}_t^i = 0$. For a detailed proof we refer to [1, Theorem 4.8]. ■

3.3.4 Asymptotic convergence when *U*o is linear

In this subsection we turn to the simpler situation where the price of the land is constant $\rho(a) \equiv \bar{\rho}$, which may be a reasonable assumption when the forest owner does not influence the land price, i.e., the available land in the market is abundant with respect to the occupied land. In this case we have a constant sale price $q = \bar{\rho} - \gamma$ and a constant buy price $p = \bar{\rho} + \gamma$ so that

$$W(a, c) = \begin{cases} -pc & c \geq 0 \\ -qc & c < 0 \end{cases}$$

We also make the assumption that a_t is always strictly less than S, as well as $\overline{a}$ which implies $r_{\overline{a}} = q$. Observe also that $p \geq r_{\underline{a}}$ with equality if $\underline{a} > 0$. We claim that there is no trading of land after time stage $2 \max_i n_i$.

Remark 3.3.7. If $c_t = 0$ after finitely many steps, then Theorem 3.3.6 holds and there is convergence to Δ^p. Notice that we recover the fixed surface problem studied in [1].

We begin with some technical results needed in the proof of our claim.

Lemma 3.3.8. *For all* $i \in I$ *we have*

(a) *if* $c_T > 0$ *and* $x^i_{T+n_i} > 0$ *then* $\bar{x}^i_{T+n_i+1} = 0$,

(b) *if* $c_T < 0$ *and* $T \geq n_i$ *then* $\bar{x}^i_{T+n_i} = 0$.

Proof. The proof is somewhat involved but the idea is simple. If there is a fraction of land occupied by trees which are not really needed, $\bar{x}^i_{T+n_i+1} > 0$ or $\bar{x}^i_{T+n_i} > 0$, we sell it at some previous stage and buy it back just in time to keep harvests unchanged, and then we compare the optimal benefit with the one obtained with the new trajectory.

(a) Assume by contradiction that $\bar{x}^i_{T+n_i+1} > 0$. Then for $\epsilon > 0$ small enough the following alternative trajectory is feasible

$$\begin{aligned} \tilde{c}_T &= c_T - \epsilon & \tilde{x}^i_{T+n_i} &= x^i_{T+n_i} - \epsilon & \tilde{\bar{x}}^i_{T+n_i+1} &= \bar{x}^i_{T+n_i+1} - \epsilon \\ \tilde{c}_{T+1} &= c_{T+1} + \epsilon & \tilde{x}^i_{T+n_i+1} &= x^i_{T+n_i+1} + \epsilon \end{aligned}$$

Depending on the sign of c_{T+1} the difference between the benefit $\tilde{V}$ of the alternative trajectory and the optimal benefit V is

$$\begin{aligned} c_{T+1} \geq 0 \;&\Rightarrow\; \tilde{V} - V = b^T \epsilon p - b^{T+1} \epsilon p = b^T (1-b) \epsilon p > 0 \\ c_{T+1} < 0 \;&\Rightarrow\; \tilde{V} - V = b^T \epsilon p - b^{T+1} \epsilon q = b^T \epsilon (p - bq) > 0 \end{aligned}$$

thus, in both cases $\tilde{V} > V$ which contradicts optimality.

(b) Assume that $\bar{x}^i_{T+n_i} > 0$. We know from the first part of the proof of Proposition 3.3.5 that there must be some $\bar{x}^i_{T+j} = 0$ with $j \in [1, n_i - 1]$. This immediately implies that there is $x^i_{T+l} > 0$ with $l \in [j, n_i - 1]$. Now, let l be the largest index in $[0, n_i - 1]$ such that $x^i_{T+l} > 0$ and consider

$$\begin{aligned} \tilde{c}_{T-n_i+l} &= c_{T-n_i+l} - \epsilon & \tilde{x}^i_{T+l} &= x^i_{T+l} - \epsilon & \tilde{\bar{x}}^i_{T+j} &= \bar{x}^i_{T+j} - \epsilon, \; j = l+1, \ldots, n_i \\ \tilde{c}_T &= c_T + \epsilon & \tilde{x}^i_{T+n_i} &= x^i_{T+n_i} + \epsilon \end{aligned}$$

Depending on the sign of c_{T-n_i+l} the difference between the benefit $\tilde{V}$ of the alternative trajectory and the optimal value V may be

$$\begin{aligned} c_{T-n_i+l} > 0 \;&\Rightarrow\; \tilde{V} - V = b^{T-n_i+l} \epsilon p - b^T \epsilon q = b^{T-n_i+l}(p - b^{n_i-l} q)\epsilon > 0 \\ c_{T-n_i+l} \leq 0 \;&\Rightarrow\; \tilde{V} - V = b^{T-n_i+l} \epsilon q - b^T \epsilon q = b^{T-n_i+l}(1 - b^{n_i-l})\epsilon q > 0. \end{aligned}$$

In both cases $\tilde{V} > V$ which contradicts optimality. ■

Proposition 3.3.9. *For all $i \in I$ we have*

(a) *if $c_T > 0$ then $x^{*i}_{\underline{a}} \geq x^i_{T+n_i}$,*

(b) *if $c_T < 0$ and $T \geq n_i$ then $x^{*i}_{\overline{a}} \leq x^i_{T+n_i}$.*

Proof. (a) The statement is trivial if $x^i_{T+n_i} = 0$. If $x^i_{T+n_i} > 0$ we consider an alternative trajectory that postpones n_i periods the purchase of an ϵ of land

$$\begin{aligned} \tilde{c}_T &= c_T - \epsilon & \tilde{x}^i_{T+n_i} &= x^i_{T+n_i} - \epsilon \\ \tilde{c}_{T+n_i} &= c_{T+n_i} + \epsilon & \tilde{u}^i_{T+n_i} &= u^i_{T+n_i} - \epsilon \end{aligned}$$

From Lemma 3.3.8 (a) we know that $u^i_{T+n_i} = \bar{x}^i_{T+n_i} + x^i_{T+n_i} \geq x^i_{T+n_i}$, hence it is enough to take $0 < \epsilon < x^i_{T+n_i}$ to assure feasibility of the proposed trajectory and we get

$$\begin{array}{ll} \text{if } c_{T+n_i} \geq 0 \;\; \Rightarrow & 0 \geq \frac{\tilde{V}-V}{b^T} = \epsilon p(1-b^{n_i}) + b^{n_i}\left[U_i(u^i_{T+n_i}-\epsilon) - U_i(u^i_{T+n_i})\right] \\ \text{if } c_{T+n_i} < 0 \;\; \Rightarrow & 0 \geq \frac{\tilde{V}-V}{b^T} = \epsilon p + b^{n_i}\left[-\epsilon q + U_i(u^i_{T+n_i}-\epsilon) - U_i(u^i_{T+n_i})\right] \\ & \qquad > \epsilon p(1-b^{n_i}) + b^{n_i}\left[U_i(u^i_{T+n_i}-\epsilon) - U_i(u^i_{T+n_i})\right] \end{array}$$

Dividing by $\epsilon > 0$ and letting $\epsilon \to 0$ we get $\sigma_i U_i'(u^i_{T+n_i}) \geq p \geq r_{\underline{a}} \geq \sigma_i U_i'(x^{*i}_{\underline{a}})$ which leads to $x^{*i}_{\underline{a}} \geq u^i_{T+n_i}$ and the proof follows because $u^i_{T+n_i} \geq x^i_{T+n_i}$.

(b) Here, the statement is trivial for $i \notin I^*_{\overline{a}}$. We consider an alternative trajectory that postpones the selling of the land. The proof is almost identical to the proof of (a), taking

$$\begin{array}{rclcrcl} \tilde{c}_T & = & c_T + \epsilon & \qquad & \tilde{x}^i_{T+n_i} & = & x^i_{T+n_i} + \epsilon \\ \tilde{c}_{T+n_i} & = & c_{T+n_i} - \epsilon & & \tilde{u}^i_{T+n_i} & = & u^i_{T+n_i} + \epsilon \end{array}$$

We conclude easily that $\sigma_i U_i'(u^i_{T+n_i}) \leq q = r_{\overline{a}} = \sigma_i U_i'(x^{*i}_{\overline{a}})$ for all $i \in I^*_{\overline{a}}$ that shows $x^{*i}_{\overline{a}} \leq u^i_{T+n_i}$ and the proposition follows from Lemma 3.3.8 (b). ■

Proposition 3.3.10. *For all $i \in I$ we have*

(a) *if $c_T > 0$ and $T \geq n_i$ then $u^i_T \geq x^{*i}_{\underline{a}}$,*

(b) *if $c_T < 0$ and $T \geq 2n_i$ then $u^i_T \leq x^{*i}_{\overline{a}}$.*

Proof. (a) The proposition is evident when $\underline{a} = 0$. If $\underline{a} > 0$, consider an alternative trajectory that brings forward the buying of an ϵ of land

(3.20)
$$\begin{array}{rclcrcl} \tilde{c}_{T-n_i} & = & c_{T-n_i} + \epsilon & \qquad & \tilde{x}^i_T & = & x^i_T + \epsilon \\ \tilde{c}_T & = & c_T - \epsilon & & \tilde{u}^i_T & = & u^i_T + \epsilon \end{array}$$

and the benefit difference fulfills

$$0 \geq \frac{\tilde{V}-V}{b^T} \geq -b^{-n_i}\epsilon p + U_i(u^i_T + \epsilon) - U_i(u^i_T) + \epsilon p$$

Dividing by $\epsilon > 0$ and letting $\epsilon \to 0$ we get $p \geq \sigma_i U_i'(u^i_T)$ which implies $x^{*i}_{\underline{a}} \leq u^i_T$ since $p = r_{\underline{a}}$.

(b) The statement is trivial if $u^i_T = 0$. If $u^i_T > 0$ and $T \geq 2n_i$, there must be $l \in [1, n_i]$ such that $x^i_{T-n_i+l} > 0$ as follows analogously to the proof of Lemma 3.3.8 (b). Let l be the greatest index in $[1, n_i]$ such that $x^i_{T-n_i+l} > 0$ and take

(3.21)
$$\begin{array}{rclcrcl} \tilde{c}_{T-2n_i+l} & = & c_{T-2n_i+l} - \epsilon, & \qquad & \tilde{x}^i_{T-n_i+l} & = & x^i_{T-n_i+l} - \epsilon \\ \tilde{\bar{x}}^i_{T-n_i+j} & = & \bar{x}^i_{T-n_i+j} - \epsilon, \;\; j = l, \ldots, n_i - 1 & & & & \\ \tilde{c}_T & = & c_T + \epsilon, & & \tilde{u}^i_T & = & u^i_T - \epsilon \end{array}$$

and again the proof follows the same lines of (a). ■

Combining all these results we conclude easily that after $2\max_i n_i$ periods there is no buying nor selling of land. That is to say, along an optimal trajectory all the transactions of land should occur during an initial period of length at most $2\max_i n_i$.

Theorem 3.3.11. *Along any optimal trajectory, $c_t = 0$ for all $t \geq 2\max_i n_i$.*

Proof. Suppose first that there is $T \geq \max_i n_i$ such that $c_T > 0$. By Proposition (3.3.10)(a) we know that $u^i_T \geq x^{*i}_{\underline{a}}$ for all i. The balance of area at stage T tells us

$$\textstyle\sum_i x^i_{T+n_i} = \sum_i u^i_T + c_T > \sum_i x^{*i}_{\underline{a}}$$

while Proposition (3.3.9)(a) gives $x^i_{T+n_i} \leq x^{*i}_{\underline{a}}$ for all i which gives a contradiction.

An similar procedure tells us that having $c_T < 0$ with $T \geq 2\max n_i$ is also absurd. ■

3.3.5 Zero transaction costs

A particularly simple situation occurs when there are no transaction costs, that is to say $\gamma = 0$. In this setting it follows easily that there is a unique area $a^* = \underline{a} = \overline{a}$ which gives a sustainable state $\Delta^* = \{\mathbb{X}^*_{a^*}\}$. Moreover, if $a^* < S$ then $\mathbb{X}_t$ converges towards $\mathbb{X}^*_{a^*}$ since $\Delta^p = \{\mathbb{X}^*_{a^*}\}$. The latter follows since conditions (3.6)(c) and (d) imply $\sigma_0 U_0'(x^0) = \sigma_i U_i'(x^i_t)$ for all $x^i_t > 0$ when $a^* \in (0, S)$ and we already know that $\Delta^p(0) = \{\mathbb{X}^*_0\}$ [3].

This property holds for every continuous and non-decreasing function ρ.

Linear land price and zero transaction costs

If in particular we have a constant function $\rho(a) = \bar{\rho}$ and no transaction costs $\gamma = 0$, $(p = q = \bar{\rho})$ we can prove that convergence to the sustainable state occurs in finite time and therefore the problem becomes finite dimensional and the optimal trajectory can be trivially found.

Proposition 3.3.12. *For each optimal trajectory we have $x^i_t = u^i_t = x^{*i}_{a^*}$ for all $t \geq 3\max_i n_i$.*

[3] An alternative way to reach this conclusion is to observe that $P(\mathbb{X}_0)$ can be seen as an instance of the problem studied in [1], by interpreting the unused land as another species with a differentiable benefit function. Under this interpretation the condition $a^* < S$ means that species 0 is present at the sustainable state and, since its maturity age is equal to one, the results in [1] imply the convergence to the sustainable state as well as the greediness of the optimal trajectory for all $t \geq 2N$.

Proof. We start by characterizing the optimal harvest: we claim that $u_t^i = x_{a^*}^{*i}$ for $t \geq 2n_i$. To prove this claim, it suffices to repeat the proof of Proposition 3.3.10. Notice that since $p = q$ no conditions are required on c_T. Using the trajectory (3.20) we get

$$0 \geq \frac{\tilde{V}-V}{b^T} = (1 - b^{-n_i})\epsilon\bar{\rho} + U_i(u_T^i + \epsilon) - U_i(u_T^i) \Rightarrow \bar{\rho} \geq \sigma_i U_i'(u_T^i) \iff x_{a^*}^{*i} \leq u_T^i$$

while the trajectory (3.21) gives us

$$0 \geq \frac{\tilde{V}-V}{b^T} = (1 - b^{-2n_i+l})\epsilon\bar{\rho} + U_i(u_T^i - \epsilon) - U_i(u_T^i) \Rightarrow \bar{\rho} \leq \sigma_i U_i'(u_T^i) \iff x_{a^*}^{*i} \geq u_T^i$$

thus we conclude $u_t^i = x_{a^*}^{*i}$ for all $t \geq 2n_i$.

From now on we take $t \geq 2\max_i n_i$. Adding up (3.2) for $i \in I$ we get

$$\textstyle\sum_{i\in I} \bar{x}_{t+n_i+1}^i = \sum_{i\in I} \bar{x}_{t+n_i}^i + \sum_{i\in I} x_{t+n_i}^i - \sum_{i\in I} u_{t+n_i}^i$$

so that using the area constraint (3.3) and remembering that $c_t = 0$ we can deduce

$$\textstyle\sum_{i\in I} \bar{x}_{t+n_i+1}^i = \sum_{i\in I} \bar{x}_{t+n_i}^i + \sum_{i\in I} u_t^i - \sum_{i\in I} u_{t+n_i}^i$$

and since $u_t^i = x_{a^*}^{*i} = u_{t+n_i}^i$ we get $\sum_{i\in I} \bar{x}_{t+n_i+1}^i = \sum_{i\in I} \bar{x}_{t+n_i}^i$. However, we know from Theorem 3.3.6 that $\bar{x}_t^i \to 0$, so the only possibility is that $\sum_{i\in I} \bar{x}_{t+n_i}^i = 0$ for all $t \geq 2\max_i n_i$, which then gives $\bar{x}_t^i = 0$ for $t \geq 3\max_i n_i$. Finally, this implies $x_t^i = u_t^i = x_{a^*}^{*i}$ for $t \geq 3\max_i n_i$ as was to be proved. ■

3.4 Land conversion costs

The results presented up to now can be applied to a broader class of problems, like those of the forests where land conversion is costly. In our case, this cost is $\gamma|x_t^0 - x_{t+1}^0|$ when we buy or sell land, that is to say when we change the use of the land from unused to a forestry use or back. In this section we aim to extend these results to the case where the land conversion from any species to another could be costly. This problem was studied by Salo and Tahvonen [33] for the case of a one species forest and an alternative annual use of the land. They showed that when land conversion costs are introduced, new optimal periodic cycles appear. This suggests that taking them into consideration will make our set Δ^p bigger, and that there may be more sustainable states. We model the conversion costs as a function g that depends only on harvests and sows at each stage t, and fulfills the following two properties

$$\text{(3.22)} \qquad \begin{array}{rl} i) & g(u_t^1, \ldots, u_t^k, x_{t+n_1}^1, \ldots, x_{t+n_k}^k) \geq 0 \\ ii) & g(u_t^1, \ldots, u_t^k, u_t^1, \ldots, u_t^k) = 0 \end{array}$$

which say that conversion costs are always non-negative and are zero when there is no land conversion. The problem is now

$$P_g(\mathbb{X}_0)\quad \begin{cases} \text{maximize} & \sum_{t=0}^{\infty} b^t \big[\sum_{i\in I} U_i(u_t^i) - g\big((u_t^i)_{i\in I}, (x_{t+n_i}^i)_{i\in I}\big)\big] \\ \text{subject to} & (3.2) \text{ and } (3.3) \\ & \boldsymbol{x}^i, \bar{\boldsymbol{x}}^i, \boldsymbol{u}^i \in \ell_+^\infty \text{ with } \mathbb{X}_0 \text{ given.} \end{cases}$$

The characterization of Δ^*, Δ^p and Δ^g is more complicated than before, depending strongly on the particular form of the function g. However, it is worth mentioning that under conditions (3.22) the function Φ defined by (3.14) where $G(\mathbb{X}_0)$ stands for the optimal greedy value of problem $P_g(\mathbb{X}_0)$ remains a Lyapunov function.

Theorem 3.4.1. *Let N be the least common multiple of $\{n_i, i \in I\}$. If $\mathbb{X}_0 \in \Delta^g$ then*

$$\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0) + \sum_{j=0}^{N-1} g(u_j^1, \ldots, u_j^k, x_{j+n_1}^1, \ldots, x_{j+n_k}^k)$$

with strict inequality unless $\mathbb{X}_0 \in \Delta^p$, so that Φ is a Lyapunov function modulo N.

Proof. We can repeat the steps of the proof of Theorem (3.14), where the Bellman's principle of dynamic programming is now stated as

$$\begin{aligned} G_0 &= \sum_{j=0}^{t-1} b^j \big[\sum_{i=0}^{k} U_j^i - g(u_j^1, \ldots, u_j^k, x_{j+n_1}^1, \ldots, x_{j+n_k}^k)\big] + b^t G_t \\ G_t &= \sum_{j=t}^{N-1} b^{j-t} \big[\sum_{i=0}^{k} U_j^i - g(u_j^1, \ldots, u_j^k, x_{j+n_1}^1, \ldots, x_{j+n_k}^k)\big] + b^{N-t} G_N \end{aligned}$$

We continue in the same manner and instead of (3.16) we retrieve

$$\Phi(\mathbb{X}_N) \geq \Phi(\mathbb{X}_0) + \sum_{j=0}^{N-1} g(u_j^1, \ldots, u_j^k, x_{j+n_1}^1, \ldots, x_{j+n_k}^k)$$

the second term of the rhs is always positive due to i), so cancelling out we conclude as before. ■

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Chapter 4

Optimal control of renewable resources with alternative use

The first part of this chapter (Sections 4.1-4.4) corresponds to the article *Optimal control of renewable resources with alternative use* [3], submitted. The second part (Sections 4.5-4.7) comprises a generalization of the results presented in the submitted article.

Abstract

We study the optimal harvesting of a renewable resource that cannot be continuously exploited, i.e., after the harvest the released space cannot be immediately reallocated to the resource. In the meantime, this space can be given an alternative use with positive utility. We consider a discrete time model over an infinite horizon with discounting. We use two different formulations of the problem to characterize the value function and the optimal harvesting policy, with the help of dynamic programming and mathematical programming.

Key words. Resource allocation, dynamic programming, mathematical programming

AMS subject classification. 93C55, 93D20, 90C39, 91B76, 91B62, 90B50

Introduction

We consider a discrete time model for the optimal harvesting of a renewable resource (land under cultivation, aqua-cultures, forests,...) on a maximal space occupancy $S > 0$. The resource can only be harvested after reaching a stage called "mature age". In the general case this mature age could be arbitrary but in this work we will consider the mature age being equal to one time step.

We assume in this study that the harvested space has to wait for at least one period before becoming available to be reallocated again to the resource. This particular feature occurs from practical constraints such as land refreshing, cleaning, or other resting duties that do not allow an immediate re-use of the space for the resource.

From the management point of view, unallocated space usually means economic loss, but we consider here that the unallocated space has a positive utility brought by an alternative use of the space compatible with the resting duties (such as tourism, scientific investigations, experimentations,...). We also assume that this alternative use can be extended to more than one time step without any a priori time limit. The management problem is then to determine at each time step the best harvesting and arbitration between the resource allocation and the alternative use.

In this model we neglect the mortality of the resource in its mature age, arguing that a rational exploitation of the resource proceeds to spatial rotations so that no space stays occupied by the same unharvested resource for a long time.

To the best of our knowledge this problem has not been discussed in the literature of mathematical modelling of natural resources (*e.g.* Clark [7] and Mitra & Wan [22]). Nevertheless, some authors have tackled the problem of the optimal harvesting of a resource when there exists an alternative use of the space, with no constraints to space allocation. In [33], Salo and Tahvonen characterize the state whose optimal trajectory is constant, the so called sustainable state, and give some numerical examples that exhibit convergence to such state. In [1], Cominetti and Piazza study a multi-species forest and prove that two types of asymptotic behavior are possible, either there is convergence to the sustainable state for every initial condition, or the optimal trajectory converges towards a periodic trajectory. This means that the choice of the transient stage (i.e. the anticipation decisions) is crucial before reaching a myopic strategy. Inspired in these ideas, we study in this paper two particular myopic strategies —the *constant policy* and the *greedy periodic policy*— analyzing the conditions under which they are optimal. These particular policies are the key ingredients for our characterization of the value function and optimal trajectories in the general case.

Our approach consists in determining first the subsets of initial conditions for which those myopic strategies are optimal. In a second stage, we prove that an optimal trajectory has to reach one of these subssets in finite time. Technically, we use two equivalent formulations of the problem, based on *dynamic programming* and *mathematical programming*. Dynamic programming aims at characterizing the value function, which is the unique solution of the Bellman equation. We may find one optimal trajectory, a posteriori. With the mathematical programming approach, we find directly a feasible optimal trajectory. This could be more convenient from the management point of view but faces the problem of the non uniqueness of the optimal trajectories.

The paper is organized as follows. In §4.1 we present the optimization model to be solved, while in §4.2 we state the main results of the paper. We first present necessary and sufficient conditions that make the myopic strategies (constant and greedy policies) optimal, and then give explicitly the value function and an optimal trajectory for every possible initial condition, solving completely the problem. In section §4.3, proofs of the results are developed: in §4.3.1, a dynamic programming approach is used while in §4.3.2, the mathematical programming theory is exploited, along with convexity and duality. Finally, conclusions are dressed in §4.4.

4.1 The optimization problem

With no loss of generality we assume that S is equal to 1. For each period $t \in \mathbb{N}$ we denote $z_t \geq 0$ the fraction of space occupied by the resource. We must decide how much resource u_t to harvest

$$0 \leq u_t \leq z_t \tag{4.1}$$

and the fraction v_t of the land occupied by the alternative use that will be reallocated to the resource

$$0 \leq v_t \leq 1 - z_t. \tag{4.2}$$

The constrains on u and v can then be expressed as

$$(u_t, v_t) \in C(z_t) \quad \text{where} \quad C(z) = [0, z] \times [0, 1 - z] \tag{4.3}$$

and the dynamics of the system are simply written as

$$z_{t+1} = z_t - u_t + v_t. \tag{4.4}$$

Given an initial condition z_0, the objective to be maximized is the discounted total benefit

$$J(z_0, u, v) = \sum_{t=0}^{\infty} b^t \left[U(u_t) + W(1 - z_t) \right]$$

over all admissible controls (u, v), where $U(\cdot)$ and $W(\cdot)$ are the utilities of the harvest and the use of the unallocated space respectively.

We assume the usual hypotheses on utility functions and discount factor. Let $\mathcal{C}_{[0,1]}$ be the set of C^1 non-decreasing concave functions from $[0,1]$ to $I\!R_+$.

Assumption A0. The functions $U(\cdot)$ and $W(\cdot)$ belong to $\mathcal{C}_{[0,1]}$. The discount rate b belongs to the interval $(0,1)$.

We denote $\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{v}$ the infinite sequences of states and controls. Admissible sequences have to fulfill the dynamics (4.4) and the feasibility constrains (4.3) at any time $t \in I\!N$; and we refer to them as trajectories. Clearly these constrains imply that $0 \le z_t \le 1$ for all $t \in I\!N$ so that $\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{v}$ belong to $\ell_+^\infty = \ell_+^\infty(I\!N)$.

Then, the optimization problem can be formulated as the search of optimal sequences $\boldsymbol{z}^\star, \boldsymbol{u}^\star, \boldsymbol{v}^\star$ solution of the following mathematical program

$$P(z_0) \begin{cases} V(z_0) = & \text{maximize} \quad \sum_{t=0}^{\infty} b^t[U(u_t) + W(1-z_t)] \\ & \text{subject to (4.3) and (4.4)} \\ & \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{v} \in \ell_+^\infty \text{ with } z_0 \text{ given.} \end{cases} \tag{4.5}$$

The existence of solutions for $P(z_0)$ follows as a consequence of the Weierstrass theorem since the feasible set is $\sigma(\ell^\infty, \ell^1)-$compact and the objective function is $\sigma(\ell^\infty, \ell^1)-$upper semi continuous. For details of the proof we refer to [1, Proposition 2.1] where a similar problem without constraint (4.2) is treated. The proof remains valid with minor changes. We cannot assure the uniqueness of optimal solutions unless U and W are strictly concave, in which case we immediately get that $\boldsymbol{u}$ and $\boldsymbol{z}$ are unique, and uniqueness of $\boldsymbol{v}$ rightly follows from the constraints.

The value function $V(z_0)$ associated with this optimal control problem can also be characterized by invoking the *dynamic programming principle*. It is a standard result (see for instance [4]) that V is the unique bounded function that satisfies Bellman equation

$$V(z) = \max_{(u,v) \in C(z)} U(u) + W(1-z) + bV(z-u+v). \tag{4.6}$$

Within this approach, optimal decisions $(u^\star, v^\star)$ are sought among state feedbacks.

Remark 4.1.1. Let $\mathcal{B}[V_0]$ be the Bellman operator defined on the set of bounded functions by

$$\mathcal{B}[V_0](z) = \max_{(u,v) \in C(z)} U(u) + W(1-z) + bV_0(z-u+v).$$

The mapping $\mathcal{B}$ is a contraction and the value function V is the unique fixed point with

$$V = \mathcal{B}[V] = \lim_{k \to \infty} \mathcal{B}^k[V_0]$$

for any bounded function V_0. Notice that when V_0 is concave the same holds for $\mathcal{B}[V_0]$, and consequently the value function V is a concave function.

4.2 The optimal trajectory

In this section we fully describe one solution of $P(z_0)$ for every $z_0 \in [0,1]$. We emphasize that we do not attempt to describe completely the solution set of $P(z_0)$, but only to propose one optimal trajectory. In §4.2.1 we discuss two particular trajectories: one constant and the other periodic. They will turn out to be the asymptotic regime of the proposed optimal trajectory which becomes either constant or periodic after at most two time steps. In §4.2.2 we describe the transient stages before reaching the steady state or periodic regime. Finally, in §4.2.3 we find the value function and a feedback law that yields one optimal trajectory.

We will use the auxiliary C^1 concave function

$$\overline{U}(z) = U(z) + W(1-z)$$

which represents the maximum benefit that one can get on a single time step when the resource is z.

4.2.1 Asymptotic behavior

In this subsection we look closely at two particular policies for problem $P(z_0)$: the *greedy periodic policy* and the *constant policy*, characterizing the states for which they are optimal.

Definition 4.2.1 (greedy periodic policy)**.** It consists in harvesting all the resource available and reallocating to the resource all the space that was assigned to the alternative use

$$u_t = z_t, \qquad v_t = 1 - z_t$$

generating a cyclic trajectory of period two: $z_{t+1} = 1 - z_t$.

The benefit of the periodic trajectory generated by the greedy policy from any initial condition $z \in [0,1]$ is

$$J_G(z) = \tfrac{\overline{U}(z) + b\overline{U}(1-z)}{1-b^2}.$$

We define p to be the largest optimal solution to

$$(P_1) \quad \text{maximize}_{z\in[0,1]} \quad J_G(z)$$

and denote $\Delta^p \subseteq [0,1]$ the set of states whose value function is $V(z) = J_G(z)$, which may be characterized as follows.

Proposition 4.2.2. *Δ^p is the set of points $z \in [0,1]$ that satisfy*

$$\overline{U}'(\max(1-z,z)) \geq b\overline{U}'(\min(1-z,z)) \,. \tag{4.7}$$

The proof will be given at sections 4.3.1 and 4.3.2.

Notice that when $\overline{U}'(\frac{1}{2}) \geq 0$, we have $p \geq \frac{1}{2}$. As a consequence of Proposition 4.2.2 we deduce the following

Lemma 4.2.3. *Δ^p is nonempty exactly when $\overline{U}'(\frac{1}{2}) \geq 0$, and then the set Δ^p is the interval $[1-p,p]$. Furthermore, the following properties are fulfilled*

i. *$\overline{U}$ is non-decreasing $[0,p]$,*

ii. *J_G is non-decreasing on $[0,p]$,*

iii. *J_G is non-increasing on $[p,1]$.*

Proof. Condition (4.7) of Proposition 4.2.2 and concavity of the function $\overline{U}(\cdot)$ imply the property

$$z \in \Delta^p \Rightarrow [\min(z,1-z),\max(z,1-z)] \subset \Delta^p \tag{4.8}$$

so Δ^p is an interval that necessarily contains $\frac{1}{2}$ whenever it is not empty. Now $\frac{1}{2}$ fulfills condition (4.7) exactly when $\overline{U}'(\frac{1}{2}) \geq 0$. The definition of p and the concavity of $\overline{U}$ together with (4.8) imply $\Delta^p = [1-p,p]$. Condition (4.7) implies that the function $\overline{U}$ is non-decreasing on $[0,\frac{1}{2}]$ and from (4.7) this implies also that $\overline{U}$ is non-decreasing on $[0,p]$. Finally, notice the property

$$J_G'(z) = \overline{U}'(z) - b\overline{U}'(1-z)$$

which implies that J_G is non-decreasing on $[0,p]$ and non-increasing on $[p,1]$. ∎

We will see in §4.2.3 that if $\overline{U}'(\frac{1}{2}) < 0$, the proposed optimal trajectory remains constant after at most two steps. This motivates the definition of

Definition 4.2.4 (constant policy). It consists in harvesting and allocating always the quantity

$$u_t = v_t = \min(z_t, 1-z_t)$$

so that the trajectory is constant: $z_t = z_0$.

The benefit of the constant trajectory from the initial condition $z \in [0,1]$ is given by

$$J_S(z) = \frac{U(\min(z,1-z))+W(1-z)}{1-b}.$$

Notice that $J_S(z) = \overline{U}(z)/(1-b)$ whenever $z \leq \frac{1}{2}$. This motivates the notion of a sustainable state, which corresponds intuitively to a state where it is optimal to stay forever.

Definition 4.2.5 (sustainable state)**.** A state $z \in [0,1]$ is called sustainable if it is invariant under the optimal policy, i.e. the constant trajectory is optimal and $V(z) = J_S(z)$.

Proposition 4.2.6. *The constant policy is optimal exactly at points z solutions to problem*

$$(P_2) \quad \textit{maximize}_{z\in[0,1/2]} \quad J_S(z)$$

Proofs will be given at sections 4.3.1 and 4.3.2. We define z^* to be the largest solution to (P_2).

4.2.2 Transient behavior

We claim that after two time steps the proposed optimal trajectory becomes either constant or periodic with period two, depending on the sign of $\overline{U}'(\frac{1}{2})$. Namely, if $\overline{U}'(\frac{1}{2}) < 0$ then after at most two steps the proposed optimal trajectory reaches z^* and remains there afterwards. Otherwise, if $\overline{U}'(\frac{1}{2}) \geq 0$ it becomes 2-periodic after two time steps. More precisely, there are two different situations depending on the initial condition. If $z_0 \in [1-p, p]$ the optimal trajectory is a 2-periodic cycle from the beginning: $(z_0, 1-z_0, z_0, \dots)$. If $z_0 \notin [1-p, p]$, then the optimal trajectory reaches the $(p, 1-p)$-cycle in one or two steps.

The previous subsection described the conditions under which the constant and greedy trajectories are optimal, from the first stage. In this subsection we discuss what happens when this is not the case, i.e., we describe precisely the behavior of the optimal trajectories during the two-step transient period before entering the periodic or steady-state regime.

We claim that the optimal policy during the transient is characterized by the auxiliary problem

$$Q(z_0)\quad \begin{cases} \text{maximize} & U(u_0) + b\overline{U}(z_1) \\ \text{subject to} & u_0 + z_1 \leq 1 \\ & 0 \leq u_0 \leq z_0 \\ & 0 \leq z_1 \leq p \end{cases}$$

This problem maximizes the two first steps benefit. The set of constrains is induced by (4.3) and (4.4) for stage $t = 0$ except for $z_1 \leq p$. This last condition was introduced to assure that the two proposed asymptotic regimes are admissible after the transient. To see this, notice that $z^* \leq p$ if and only if $\overline{U}'(\frac{1}{2}) \leq 0$.

For convenience we define the function

$$Q(z) = U(z) + b\overline{U}(1-z),\ z \in [0,1]$$

and denote by $q \in [0,1]$ the point where it attains the maximum. If this point is not unique we take q as the largest optimal solution.

Problem $Q(z_0)$ can be solved explicitly: when $\overline{U}'(\frac{1}{2}) \leq 0$ one optimal solution is given by

$$(z_1, u_0) = \begin{cases} (z^*, z_0) & \text{when } z_0 \in [0, 1-z^*) \\ (1-z_0, z_0) & \text{when } z_0 \in [1-z^*, q] \\ (1-q, q) & \text{when } z_0 \in (q, 1] \end{cases}$$

while if $\overline{U}'(\frac{1}{2}) > 0$ we have

$$(z_1, u_0) = \begin{cases} (p, z_0) & \text{when } z_0 \in [0, 1-p) \\ (1-z_0, z_0) & \text{when } z_0 \in (1-p, q] \\ (1-q, q) & \text{when } z_0 \in (q, 1] \end{cases}$$

4.2.3 The optimal policy

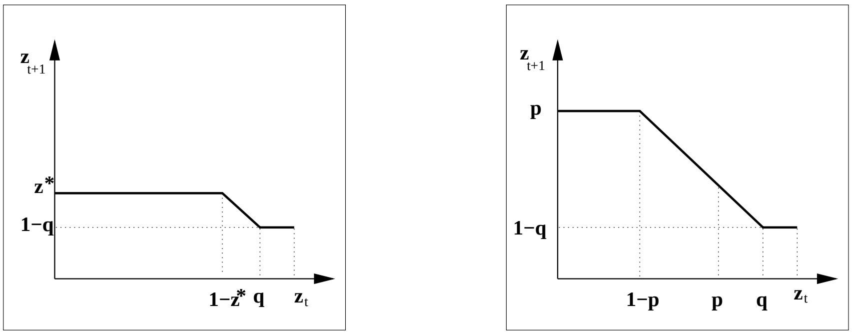
Using the results of the previous sections we may find explicitly the value function and the feedback law that gives the value z_{t+1} as a function of z_t for all t. We distinguish three different situations according to the position of the solution z^* of (P_1) in the interval $[0, \frac{1}{2}]$: Left, Interior, or Right.

$$\begin{array}{ll} (\mathbf{S_L}) & z^* = 0, \quad \overline{U}'(0) \leq 0 \\ (\mathbf{S_I}) & z^* \in (0, \frac{1}{2}), \quad \overline{U}'(z^*) = 0 \\ (\mathbf{S_R}) & z^* = \frac{1}{2}, \quad \overline{U}'(\frac{1}{2}) \geq 0 \end{array}$$

The graph of the feedback law is presented in Figure 4.1. To the left, we see the feedback law when $(\mathbf{S_I})$ holds and to the right when $(\mathbf{S_R})$ holds. The case $(\mathbf{S_L})$ can be seen as a degenerate case of the former where we simply have $z_t \equiv 0$ for all $t \geq 1$. In all situations it may happen that $q = 1$, in which case the last interval disappears. The graphs are very similar and could be merged in a single diagram, but we prefer to distinguish them since they give rise to different dynamics.

We sometimes refer to $\boldsymbol{z}$ as the optimal trajectory, understanding that the optimal $\boldsymbol{u}$ and $\boldsymbol{v}$ are easily deduced from $\boldsymbol{z}$ as follows

$$\begin{aligned} u_t &= \min(z_t, 1 - z_{t+1}), \\ v_t &= \min(1 - z_t, z_{t+1}). \end{aligned} \tag{4.9}$$



(a) (b)

Figure 4.1: Feedback law.

In the mathematical programming approach we consider $\boldsymbol{u}$ and $\boldsymbol{v}$ as variables linked to $\boldsymbol{z}$ by the feasibility constrains (4.3) and (4.4). Although this increases the number of variables, the proofs become somewhat simpler as we avoid dealing with the non-differentiable function $\min(\cdot,\cdot)$.

In Figure 4.2 we present the optimal trajectory from a state $z_0 \in (p,q]$ when $(\mathbf{S_R})$ holds. Let us observe, that the trajectory reaches the periodic cycle $p, 1-p, p, \dots$ in two steps, the transient being $z_1 = 1 - z_0$. The evolution from any other initial state can be found similarly. We remark that for every initial state, the optimal trajectory reaches the $(p, 1-p)-$cycle in one or two steps, except for the initial conditions in the interval $z \in [1-p, p]$, whose optimal trajectory is the periodic cycle $z, 1-z, z, \dots$ from the first stage.

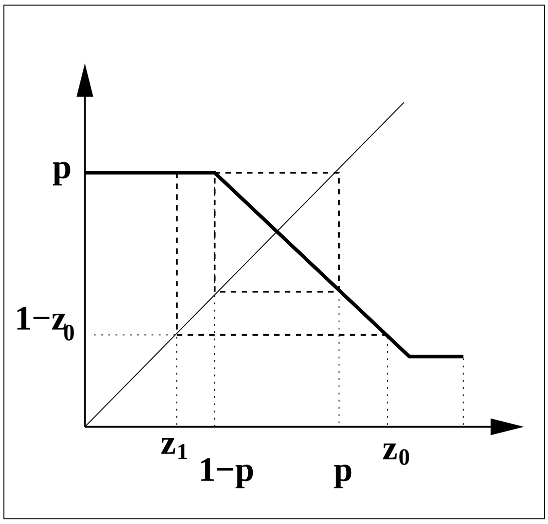


Figure 4.2: Dynamics

If $(\mathbf{S_I})$ holds the dynamics are even simpler and it is not difficult to see that there is convergence to the state z^* in one or two steps.

The following theorem describes more precisely the feedback law represented in Figure 4.1 and proves that it is effectively optimal.

Theorem 4.2.7. *If* $(\mathbf{S_L})$ *or* $(\mathbf{S_I})$ *holds there is convergence in two steps to a sustainable state and the value function is*

$$V(z) = \begin{cases} \overline{U}(z) + bJ_S(z^*), & z \in [0, 1-z^*] \\ W(1-z) + \max\limits_{z' \in [1-z^*, z]} Q(z') + b^2 J_S(z^*), & z \in [1-z^*, 1] \end{cases} \tag{4.10}$$

which is attained by the optimal trajectory generated by the feedback law of Figure 4.1(a),

$$\begin{cases} (z_0, z^*, z^*, \dots) & \textit{when } z_0 \in [0, 1-z^*) \\ (z_0, 1-z_0, z^*, \dots) & \textit{when } z_0 \in [1-z^*, q] \\ (z_0, q, z^*, \dots) & \textit{when } z_0 \in (q, 1] \end{cases} \tag{4.11}$$

Otherwise, if $(\mathbf{S_R})$ *holds then* Δ^p *is reached in two steps and the optimal trajectory is greedy of period two afterwards, the value function being*

$$V(z) = \begin{cases} \overline{U}(z) + bJ_G(p), & z \in [0, 1-p) \\ J_G(z), & z \in [1-p, p] \\ W(1-z) + \max\limits_{z' \in [p, z]} Q(z') + b^2 J_G(p), & z \in (p, 1] \end{cases} \tag{4.12}$$

which is attained by the optimal trajectory generated by the feedback law of Figure 4.1(b),

$$\begin{cases} (z_0, p, 1-p, p, 1-p, \dots) & \textit{when } z_0 \in [0, 1-p) \\ (z_0, 1-z_0, z_0, 1-z_0, \dots) & \textit{when } z_0 \in [1-p, p] \\ (z_0, 1-z_0, p, 1-p, p, \dots) & \textit{when } z_0 \in (p, q] \\ (z_0, 1-q, p, 1-p, p, \dots) & \textit{when } z_0 \in (q, 1] \end{cases} \tag{4.13}$$

4.3 Proofs

The proofs of Propositions 4.2.2 and 4.2.6 and Theorem 4.2.7 are given in this section with two different approaches.

Let us first denote by $\mathcal{U}(z, z')$ the set of feasible controls, that induce the transition of the system from state z to z'

$$\mathcal{U}(z, z') = \{(u, v) \in C(z) \mid z' = z - u + v\},$$

and notice that $\mathcal{U}(z,z')$ is never empty for any pair $(z,z') \in [0,1]^2$. Looking carefully to the definition of the set $\mathcal{U}$ we see that there is an even simpler expression

$$\mathcal{U}(z,z') = \big\{(u,v) \,|\, v = z' - z + u,\ u \in [\max(0, z - z'), \min(z, 1 - z')]\big\}$$

Then, the dynamic programming equation (4.6) can take the simpler expression

$$\begin{aligned} V(z) &= \max_{z' \in [0,1]} \max_{(u,v) \in \mathcal{U}(z,z')} U(u) + W(1-z) + bV(z') \\ &= W(1-z) + \max_{z' \in [0,1]} \max_{u \in \mathcal{U}(z,z')} U(u) + bV(z') \\ &= W(1-z) + \max_{z' \in [0,1]} U(\min(z, 1-z')) + bV(z') \end{aligned} \tag{4.14}$$

4.3.1 Dynamic programming proofs

Optimality of the greedy policy

Proof of Proposition 4.2.2 Denote by $\phi(\cdot)$ the function

$$\phi(\xi, \xi') = U(\min(\xi, 1 - \xi')) + bJ_G[\overline{U}](\xi') \, .$$

If the greedy policy is optimal at z, then necessarily one has $V(z) = J_G[\overline{U}](z)$ and $V(1 - z) = J_G[\overline{U}](1 - z)$. From equation (4.14), this is equivalent to write that the functions $\xi' \mapsto \phi(z, \xi')$ and $\xi' \mapsto \phi(1 - z, \xi')$ are maximized respectively at $\xi' = 1 - z$ and $\xi' = z$. Functions $\xi' \mapsto \phi(\xi, \xi')$ being concave (by composition of concave functions), this is also equivalent to require

$$0 \in \partial_2^+ \phi(\xi, 1 - \xi) \, , \quad \xi \in \{z, 1 - z\} \, , \tag{4.15}$$

where $\partial_2^+ \phi$ denotes the Fréchet super-differential of ϕ w.r.t. to its second argument. Simple calculations give

$$\partial_2^+ \phi(\xi, 1 - \xi) = [-U'(\xi), 0] + bJ_G[\overline{U}]'(1 - \xi)$$

and

$$J_G[\overline{U}]'(1 - \xi) = \frac{\overline{U}'(1 - \xi) - b\overline{U}'(\xi)}{1 - b^2} \, .$$

Then, conditions (4.15) can be rewritten as

$$\begin{cases} \overline{U}'(1 - \xi) \geq b\overline{U}'(\xi) \\ U'(\xi) - b^2 W'(1 - \xi) \geq b\overline{U}'(1 - \xi) \end{cases} \quad \xi \in \{z, 1 - z\} \, ,$$

Noticing that $U'(\xi) - b^2 W'(1 - \xi) \geq \overline{U}'(\xi)$, these last conditions are equivalent to the simpler ones

$$\overline{U}'(1 - \xi) \geq b\overline{U}'(\xi) \, , \quad \xi \in \{z, 1 - z\} \, .$$

Finally, $\overline{U}(\cdot)$ being concave, we obtain exactly the single condition (4.7).

Conversely, assume that condition (4.7) holds at a given z. Denote $p = \min(z, 1-z)$ and notice that (4.7) is also fulfilled at any $z \in [p, 1-p]$, because $\overline{U}(\cdot)$ is concave. Consider then the operator $\mathcal{M}_p : \mathcal{C}_{[0,1]} \to \mathcal{C}_{[0,1]}$ defined as follows

$$\mathcal{M}_p[f](\xi) = \begin{cases} f(1-p) + f'(1-p)(\xi - 1 + p), & \xi \in [0, 1-p] \\ f(\xi), & \xi \in [1-p, p] \\ f(p) + f'(p)(\xi - p), & \xi \in [p, 1] \end{cases}$$

and notice that $\mathcal{M}_p[f] \geq f$ for any $f \in \mathcal{C}_{[0,1]}$. Let $\widetilde{V}$ be the value function for the modified utility functions $\widetilde{U} = \mathcal{M}_p[U]$, $\widetilde{W} = \mathcal{M}_p[W]$ and define the function $\overline{\widetilde{U}}(z) = \widetilde{U}(z) + \widetilde{W}(1-z)$. One has clearly $\widetilde{V} \geq V \geq J_G[\overline{U}]$, and the function $J_G[\overline{\widetilde{U}}]$ is solution of the Bellman equation (4.14), because condition (4.7) is fulfilled for $\overline{\widetilde{U}}$ at any $z \in [0, 1]$. Consequently, one has $\widetilde{V}(z) = J_G[\overline{\widetilde{U}}]$. Finally, remark that $J_G[\overline{U}] = J_G[\overline{\widetilde{U}}]$ on $[1-p, p]$, which proves that $V = J_G[\overline{U}]$ on $[1-p, p]$. The greedy policy is then optimal from any $z \in [1-p, p]$. ■

Optimality of the constant policy

Proof of Proposition 4.2.6 Let us first show that the constant policy cannot be optimal for $z > \frac{1}{2}$. If it is true, one has $\overline{U}(z), \overline{U}(1-z) > U(1-z) + W(1-z)$ and consequently

$$J^G[\overline{U}](z) > \frac{(U(1-z) + W(1-z))(1+b)}{1-b^2} = J^S(z) ,$$

thus a contradiction with the optimality of the constant policy.

If the constant policy is optimal at $z \in [0, \frac{1}{2}]$, the following inequality is obtained from the Bellman equation (4.14)

$$\begin{aligned} V(z) = J^S(z) &\geq W(1-z) + U(\min(z, 1-z^*)) + bJ^S(z^*) \\ &= U(z) + W(1-z) + bJ^S(z^*) = J^S(z)(1-b) + bJ^S(z^*) \end{aligned}$$

from which we deduce $J^S(z) \geq J^S(z^*)$ i.e. $\overline{U}(z) \geq \overline{U}(z^*)$. So the constant policy cannot be optimal away from z^*.

If $z^* = \frac{1}{2}$, one has necessarily $\overline{U}'(\frac{1}{2}) \geq 0$ and from Proposition 1, the constant policy that coincides with the greedy policy at $z^* = \frac{1}{2}$ is optimal. Otherwise, z^* is a maximizer of the concave function $\overline{U}$ on $[0, 1]$. From the Bellman equation (4.14), one can write the following inequalities

$$\begin{aligned} V(z) &= \overline{U}(z) + \max_{z'\in[0,1]} U(\min(z,1-z')) - U(z) + bV(z') \\ &\leq \overline{U}(z^*) + b\max_{z'\in[0,1]} V(z') = J^S(z^*)(1-b) + b\max_{z'\in[0,1]} V(z') \end{aligned}$$

from which we deduce $V(z^*) = J^S(z^*) = \max_{z'\in[0,1]} V(z')$. The constant policy is then optimal from z^*. ■

Optimal policy

Let $V(\cdot)$ be a bounded function defined on $[0,1]$. Denoting

$$\Delta V(z,z') = W(1-z) + U(\min(z,1-z')) + bV(z')$$

$V(\cdot)$ is the value function if it is solution of the Bellman equation (4.14), which amounts to prove that the equality

$$\max_{z'\in[0,1]} \Delta V(z,z') = V(z) \qquad (4.16)$$

is fulfilled for any $z \in [0,1]$.

Proof of Theorem 4.2.7. We claim that if $(\mathbf{S_R})$ holds, the value function is (4.12). To see this, we show that condition (4.16) is fulfilled for any $z \in [0,1]$. We distinguish three cases depending on z belonging to $\Delta^p = [1-p,p]$ or not.

1. $z < 1-p$. When $z' < 1-p$, one has

$$\begin{aligned} \Delta V(z,z') &= \overline{U}(z) + b\overline{U}(z') + b^2 J_G(p) \\ &\leq \overline{U}(z) + b\overline{U}(1-p) + b^2 J_G(p) && \text{(by Lemma 4.2.3.i)} \\ &= \overline{U}(z) + bJ_G(1-p) \\ &\leq \overline{U}(z) + bJ_G(p) = V(z) && \text{(by Lemma 4.2.3.ii)} \end{aligned}$$

   For $z' \in [1-p,p]$, we can write using Lemma 4.2.3.ii)

$$\Delta V(z,z') = \overline{U}(z) + bJ_G(z') \leq \overline{U}(z) + bJ_G(p) = V(z)$$

   Finally, for $z' > p$,

$$\begin{aligned} \Delta V(z,z') &\leq \overline{U}(z) + b\left[W(1-z') + \max\nolimits_{z''\in[p,z']} Q(z'')\right] + b^3 J_G(p) \\ &\leq \overline{U}(z) + b\left[W(1-z') + U(z') + b\overline{U}(1-p)\right] + b^3 J_G(p) \\ &\qquad \text{(by Lemma 4.2.3.i.)} \\ &\leq \overline{U}(z) + b\left[\overline{U}(p) + b\overline{U}(1-p)\right] + b^3 J_G(p) \\ &\qquad \text{(by Lemma 4.2.3.iii)} \\ &= \overline{U}(z) + bJ_G(p) = V(z) \end{aligned}$$

In any case, one has $\Delta V(z,z') \leq V(z)$ and $\Delta V(z,p) = V(z)$. Consequently, (4.16) is fulfilled.

2. $z \in [1-p, p]$. When $z' < 1-p$, one has

$$\begin{aligned}\Delta V(z,z') &= W(1-z) + U(z) + b\overline{U}(z') + b^2 J_G(p) \\ &\leq \overline{U}(z) + b\overline{U}(1-p) + b^2 J_G(p) && \text{(by Lemma 4.2.3.i)} \\ &= \overline{U}(z) + bJ_G(1-p) \\ &\leq \overline{U}(z) + bJ_G(1-z) = V(z) && \text{(by Lemma 4.2.3.ii)}\end{aligned}$$

for $z' \in [1-p, 1-z]$, using Lemma 4.2.3.ii we get

$$\Delta V(z,z') = \overline{U}(z) + bJ_G(z') \leq \overline{U}(z) + bJ_G(1-z) = V(z) ,$$

for $z' \in [1-z, p]$, using again by Lemma 4.2.3.ii

$$\begin{aligned}\Delta V(z,z') &= W(1-z) + U(1-z') + bJ_G(z') \\ &= W(1-z) + J_G(1-z') - W(z') \leq J_G(z) = V(z) ,\end{aligned}$$

and for $z' > p$,

$$\begin{aligned}\Delta V(z,z') =& \; W(1-z) + U(1-z') + b\left[W(1-z') + Q(z'')\right] + b^3 J_G(p) \\ & \qquad\qquad\qquad\qquad \text{with } z'' \in [p, z'] \\ \leq& \; W(p) + U(1-p) + b\left[W(1-z'') + U(z'')) + b\overline{U}(1-z'')\right] + b^3 J_G(p) \\ =& \; \overline{U}(1-p) + b[\overline{U}(z'') + b\overline{U}(1-z'')] + b^3 J_G(p) \\ =& \; \overline{U}(1-p) + bJ_G(z'')(1-b^2) + b^3 J_G(p) \\ \leq& \; \overline{U}(1-p) + bJ_G(p) && \text{(Lemma 4.2.3.iii)} \\ =& \; J_G(1-p) \leq J_G(z) = V(z) && \text{(Lemma 4.2.3.ii)}\end{aligned}$$

Notice also that $\Delta V(z, 1-z) = V(z)$. Finally (4.16) is fulfilled.

3. $z > p$. When $z' < 1-z$, one has (using Lemma 4.2.3.i)

$$\Delta V(z,z') \leq \overline{U}(z) + b\overline{U}(1-z) + b^2 J_G(p) = \Delta V(z, 1-z)$$

and when $z' \in [1-z, 1-p]$,

$$\Delta V(z,z') = W(1-z) + Q(z') + b^2 J_G(p) \leq V(z) .$$

One has also $\Delta V(z, \hat{q}) = V(z)$ for $\hat{q}$ that realizes the maximum in (4.12).

For $z' \in [p, 1-p]$, one can write

$$\begin{aligned}\Delta V(z,z') &= W(1-z) + U(1-z') + bJ_G(z') \\ &= W(1-z) + J_G(1-z') - W(z') \\ &\leq W(1-z) + J_G(p) - W(1-p) && \text{(by Lemma 4.2.3.ii)} \\ &= W(1-z) + U(p) + bJ_G(1-p) \\ &= W(1-z) + Q(p) + b^2 J_G(p) \leq V(z)\end{aligned}$$

and for $z' > p$,

$$\begin{aligned}\Delta V(z,z') \leq\ & W(1-z)+U(1-z')+b\left[W(1-z')+U(z')+b\overline{U}(1-p)+b^2J_G(p)\right] \\ & \text{(Lemma 4.2.3.i)} \\ =\ & W(1-z)+Q(1-z')+b^2J_G(1-p) \\ =\ & W(1-z)+J_G(1-z')(1-b^2)-W(z')+b^2J_G(1-p) \\ \leq\ & W(1-z)+J_G(p)(1-b^2)-W(p)+b^2J_G(p) \\ & \text{(Lemma 4.2.3.ii)} \\ =\ & W(1-z)+Q(p)+b^2J_G(p) \leq V(z)\end{aligned}$$

Finally $\Delta V(z,z') \leq V(z)$ for any z' and (4.16) is also fulfilled in this case.

We can easily deduce from above that if $(\mathbf{S_R})$ holds, the trajectory built by the feedback

$$(u^\star, v^\star)(z) = \begin{cases} (z,p) & \text{when } z \in [0, 1-p] \\ (z, 1-z) & \text{when } z \in [1-p, p] \\ (q_z, 1-z) & \text{when } z \in [p, 1] \end{cases}$$

(where q_z realizes the maximum of the function $Q(\cdot)$ on the interval $[p,z]$) is optimal and reaches a greedy cycle in at most two steps.

More precisely, the trajectory described in (4.11) is optimal.

We have completely solved the problem if $(\mathbf{S_R})$ holds. Let us consider what happens if $(\mathbf{S_L})$ or $(\mathbf{S_I})$ holds.

When $\overline{U}'(\frac{1}{2}) < 0$ the value function is (4.10).

Notice first that in this case z^* is strictly less than $\frac{1}{2}$.

To prove that (4.16) is fulfilled for any $z \in [0,1]$, we distinguish two cases.

1. $z < 1-z^*$. For $z' \leq 1-z^*$, one can write

   $$\Delta V(z,z') \leq W(1-z)+U(z)+b\overline{U}(z')+b^2J_S(z^*) \leq \overline{U}(z)+b\overline{U}(z^*)+b^2J_S(z^*) = V(z)\ ,$$

   and notice that the equality $\Delta V(z,z^*) = V(z)$ is fulfilled. For $z' \geq 1-z^*$, one has

   $$\begin{aligned}\Delta V(z,z') \ \leq\ & W(1-z)+U(z)+bW(1-z')+bU(z')+b^2\overline{U}(z^*)+b^3J_S(z^*) \\ =\ & \overline{U}(z)+b\overline{U}(z')+b^2J_S(z^*) \leq \overline{U}(z)+b\overline{U}(z^*)+b^2J_S(z^*) = V(z)\ .\end{aligned}$$

2. $z \geq 1-z^*$. When $z' < 1-z^*$, notice that the following properties are fulfilled

   $$\begin{cases} z' \leq 1-z & \Rightarrow\ U(\min(z,1-z'))+b\overline{U}(z') \leq U(z)+b\overline{U}(1-z) = Q(z)\ , \\ z' \geq z^* & \Rightarrow\ U(\min(z,1-z'))+b\overline{U}(z') leq U(1-z^*)+b\overline{U}(z^*) = Q(1-z^*)\ , \end{cases}$$

because $\overline{U}(\cdot)$ is non-decreasing on $[0, z^*]$ and maximized at z^*. Consequently, one has

$$\begin{aligned}\Delta V(z,z') = & \; W(1-z) + U(\min(z,1-z')) + b\overline{U}(z') + b^2 J_S(z^*) \\ \leq & \; W(1-z) + \max(Q(z), Q(1-z^*)) + b^2 J_S(z^*) \leq V(z)\ ,\end{aligned}$$

and $\Delta V(z,\hat{q}) = V(z)$, where $\hat{q}$ realizes the maximum of the function $Q(\cdot)$ over $[1-z^*, z]$. Finally, when $z' > 1 - z^*$ (if $z^* > 0$), one can write

$$\begin{aligned}\Delta V(z,z') &= W(1-z) + U(1-z') + bW(1-z') + \max_{z'' \in [1-z^*, z']} Q(z'') + b^3 J_S(z^*) \\ &\leq W(1-z) + U(1-z') + bW(1-z') + bU(z') + b^2\overline{U}(z^*) + b^3 J_S(z^*) \\ &\leq W(1-z) + Q(1-z') + b^2 J_S(z^*) \leq V(z)\ .\end{aligned}$$

In any case, we have obtained condition (4.16).

We can observe now that the trajectory given by the following optimal feedbacks

$$(u^\star, v^\star)(z) = \begin{cases} (z, z^*) & \text{when } z \in [0, 1-z^*] \\ (\hat{q}_z, 1-z) & \text{when } z \in [1-z^*, 1] \end{cases}$$

(where $\hat{q}_z$ realizes the maximum of the function $Q(\cdot)$ on the interval $[1 - z^*, z]$) is optimal and reaches the sustainable state z^* in at most two steps.

More precisely, trajectory (4.13) is optimal.

4.3.2 Mathematical programming proofs

Optimality of the greedy policy

Throughout we let $\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{v}$ be an optimal trajectory for $P(z_0)$.

Proof of Proposition 4.2.2 To prove the necessity of condition (4.7) take $z_0 \in \Delta^p$ and consider the greedy trajectory issued from z_0: $\boldsymbol{z} = (z_0, 1-z_0, z_0, 1-z_0, \dots)$ and $\boldsymbol{u} = \boldsymbol{z}$ and $\boldsymbol{v} = 1 - \boldsymbol{z}$. Suppose that $1 - z_0 > z_0$ and consider the perturbed trajectory that consists in sowing $1 - z_0 - \epsilon$ at $t = 0$ instead of $1 - z_0$, keeping this extra land in the alternative use for one period and sowing $z_0 + \epsilon$ at $t = 1$, after which we continue with a greedy periodic policy. If the original trajectory is optimal, the benefit obtained with it (V) must be greater or equal to the one obtained with the new trajectory ($\tilde{V}$), this is

$$\begin{aligned}V - \tilde{V} = & \; \tfrac{b}{1-b^2}[U(1-z_0) - U(1-z_0-\epsilon)] + \tfrac{b^2}{1-b^2}[U(z_0) - U(z_0+\epsilon)] \\ & + \tfrac{b}{1-b^2}[W(z_0) - W(z_0+\epsilon)] + \tfrac{b^2}{1-b^2}[W(1-z_0) - W(1-z_0-\epsilon)] \geq 0.\end{aligned}$$

Dividing by $\epsilon > 0$ and letting it to 0 yields (4.7). The proof in the case $z_0 > 1 - z_0$ is completely analogous, perturbing the optimal trajectory at $t = 1$ instead of $t = 0$.

To prove the sufficiency of condition (4.7) let us show that the greedy periodic trajectory is optimal for $P(z_0)$. We prove the optimality using the Karush-Kuhn-Tucker (KKT) theorem. To this end we consider the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} b^t \big[U(u_t) + W(1-z_t) + \mu_t u_t + \nu_t v_t + \theta_t(z_t - z_{t+1} - u_t + v_t) + \alpha_t(z_t - u_t) + \beta_t(1 - z_t - v_t)\big] \tag{4.17}$$

together with the following set of ℓ^1-multipliers

$$\begin{cases} \mu_t = \nu_t = 0 \\ \beta_t = \theta_t = \frac{b^{t+1}}{1-b^2}[\overline{U}'(1-z_t) - b\overline{U}'(z_t)] \\ \alpha_t = \frac{b^t}{1-b^2}[Q'(z_t) - b^2 W'(1-z_t)]. \end{cases}$$

In fact, they belong to ℓ^1_+ due to condition (4.7). We can see that we have complementary slackness and the feasibility constrains are fulfilled. To prove that $\nabla \mathcal{L} = 0$, it is left to see that $\mathcal{L}_{u_t} = \mathcal{L}_{z_t} = \mathcal{L}_{v_t} = 0 \ \forall t$.

$$\begin{aligned} \mathcal{L}_{u_t} &= b^t U'(u_t) + \mu_t - \theta_t - \alpha_t \\ &= b^t U'(z_t) - \tfrac{b^{t+1}}{1-b^2}[\overline{U}'(1-z_t) - b\overline{U}'(z_t)] - \tfrac{b^t}{1-b^2}[Q'(z_t) - b^2 W'(1-z_t)] \\ &= \tfrac{b^t}{1-b^2}[(1-b^2)U'(z_t) - b\overline{U}'(1-z_t) + b^2\overline{U}'(z_t) - U'(z_t) + b\overline{U}'(1-z_t) + b^2 W'(1-z_t)] \\ &= 0 \end{aligned} \tag{4.18}$$

$$\begin{aligned} \mathcal{L}_{z_t} &= -b^t W'(1-z_t) + \theta_t - \theta_{t-1} + \alpha_t - \beta_t \\ &= -b^t W'(1-z_t) - \tfrac{b^t}{1-b^2}[\overline{U}'(1-z_{t-1}) - b\overline{U}'(z_{t-1})] + \tfrac{b^t}{1-b^2}[Q'(z_t) - b^2 W'(1-z_t)] \\ &= \tfrac{b^t}{1-b^2}[(b^2-1)W'(1-z_t) - \overline{U}'(z_t) + b\overline{U}'(1-z_t) + U'(z_t) - b\overline{U}'(1-z_t) - b^2 W'(1-z_t)] \\ &= 0 \end{aligned} \tag{4.19}$$

$$\mathcal{L}_{v_t} = \nu_t + \theta_t - \beta_t = 0 \tag{4.20}$$

Thus, the proposed trajectory is a stationary point of the Lagrangian (4.17) and hence a solution of problem $P(z_0)$. ■

Optimality of the constant policy

Proof of Proposition 4.2.6 Let us show that the stationary trajectory $z_t = u_t = v_t = z^*$ is optimal for $P(z^*)$. To this end consider the Lagrangian (4.17) with the following set of ℓ^1_+-multipliers

$$\begin{cases} \nu_t = 0 \\ \beta_t = \theta_t = \frac{b^{t+1}}{1+b}\left[\overline{U}'(z^*)\right]_+ \\ \alpha_t = \frac{b^t}{1+b}\left[\overline{U}'(z^*)\right]_+ + b^t W'(1-z^*) \\ \mu_t = b^t\left[-\overline{U}'(z^*)\right]_+ \end{cases}$$

where $[x]_+$ stands for the positive part of x. A routine verification shows that $\nabla\mathcal{L} = 0$ and we have complementary slackness which implies that the proposed trajectory is a stationary point for $P(z^*)$. Hence, it is an optimal solution and z^* is sustainable.

To prove the other implication, let $\check{z}$ be a sustainable state. Let us show that $\check{z} \in \arg\max_{z\in[0,\frac{1}{2}]} \overline{U}(z)$. To this end, consider the following trajectory $\tilde{z} = \check{z} - \epsilon \quad \forall t \geq 1$, $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{z}}$ and $\tilde{v}_t = \tilde{z}_{t+1}$. Given that $\check{z}$ is a sustainable state, the benefit obtained with the constant trajectory must be greater or equal to the one obtained with the alternative trajectory, which is equivalent to

$$U(\check{z}) - U(\check{z} - \epsilon) + W(1 - \check{z}) - W(1 - \check{z} - \epsilon) \geq 0 \tag{4.21}$$

If $\check{z} = \frac{1}{2}$, then the proposed alternative trajectory is feasible for $0 < \epsilon < \frac{1}{2}$. So dividing (4.21) by ϵ and letting $\epsilon \to 0$ yields

$$U'(\check{z}) - W'(1 - \check{z}) \geq 0 \iff \overline{U}'(\check{z}) \geq 0$$

which readily implies that $\check{z} = \frac{1}{2} \in \arg\max_{z\in[0,\frac{1}{2}]} \overline{U}(z)$. If $\check{z} \in (0, \frac{1}{2})$ we can consider ϵ positive and negative provided that $|\epsilon|$ is small enough. So, repeating the reasoning we get

$$\overline{U}'(\check{z}) = 0$$

which is equivalent to $\check{z} \in \arg\max_{z\in[0,\frac{1}{2}]} \overline{U}(z)$. And finally, if $\check{z} = 0$, only the values of $\epsilon < 0$ can be considered and this yields $\overline{U}'(0) \leq 0$, which also gives us $\check{z} \in \arg\max_{z\in[0,\frac{1}{2}]} \overline{U}(z)$. ■

Optimal policy

Proof of Theorem 4.2.7 If $(\mathbf{S_L})$ holds we propose the primal solution $z_t = 0$ for all $t \geq 1$, $\boldsymbol{u} = \boldsymbol{v} = 0$ with the ℓ^1_+-multipliers $\beta = \theta = \nu = 0$ and

$$\begin{cases} \alpha_t = b^t W'(1) \\ \mu_t = b^t[W'(1) - U'(z_t)] \end{cases}$$

It is easy to see that the proposed trajectory is a stationary point of the Lagrangian (4.17) and thus an optimal solution of $P(z_0)$.

If $(\mathbf{S_I})$ holds, z_t evolves following the feedback law showed in Figure 4.1(a), reaching z^* in at most two steps. We recall the proposed optimal trajectory

$$(4.11) \qquad \begin{cases} (z_0, z^*, z^*, \dots) & \text{when } z_0 \in [0, 1-z^*] \\ (z_0, 1-z_0, z^*, z^*, \dots) & \text{when } z_0 \in (1-z^*, q] \\ (z_0, q, z^*, z^*, \dots) & \text{when } z_0 \in (q, 1] \end{cases}$$

We can see that u_t is simply expressed as $u_t = \min(z_t, q)$ (see Equations 4.9).

We propose the following ℓ^1_+-multipliers: $\mu = \nu = 0$, $\alpha_t = b^t\alpha(z_t)$ and $\theta_t = \beta_t = b^t\theta(z_t)$ where

$$\alpha(z) = \begin{cases} U'(z) & \text{if } z \in [0, 1-z^*] \\ Q'(z) & \text{if } z \in [1-z^*, q] \\ 0 & \text{if } z \in [q, 1] \end{cases}$$

$$\theta(z) = \begin{cases} 0 & \text{if } z \in [0, 1-z^*] \\ b\overline{U}'(1-z) & \text{if } z \in (1-z^*, q] \\ U'(q) & \text{if } z \in (q, 1] \end{cases}$$

Observe that the functions $\alpha(z)$ and $\theta(z)$ are continuous in the interval $[0, 1]$.

We proceed now to prove that $\nabla\mathcal{L} = 0$, starting with the partial derivative with respect to u_t.

$$(4.18) \Rightarrow \mathcal{L}_{u_t} = b^t[U'(u(z_t)) + 0 - \theta(z_t) - \alpha(z_t)]$$

Hence, to conclude $\mathcal{L}_{u_t} = 0$, it is enough to prove

$$(4.22) \qquad U'(u(z)) - \theta(z) - \alpha(z) = 0 \qquad \text{for all } z \in [0, 1]$$

To this end, we separate the study in three different cases

- $z \in [0, 1-z^*]:$ (4.22) $\Leftrightarrow U'(z) - 0 - U'(z) = 0$
- $z \in (1-z^*, q]:$ (4.22) $\Leftrightarrow U'(z) - b\overline{U}'(1-z) - Q'(z) = 0$
- $z \in (q, 1]:$ (4.22) $\Leftrightarrow U'(q) - U'(q) - 0 = 0$

By (4.19)

$$\mathcal{L}_{z_t} = -b^t[W(1 - z_t) - \theta(z_t) + \tfrac{1}{b}\theta(z_{t-1}) - \alpha(z_t) + \beta(z_t)]$$

Here, we have two steps of time involved which makes the proof somewhat more involved. To show that

$$(4.23) \qquad W(1 - z_t) + \tfrac{1}{b}\theta(z_{t-1}) - \alpha(z_t) = 0 \quad \forall t,$$

we divide again the study into three cases

- $z_{t-1} \in [0, 1-z^*)$: $z_t = z^*$ and
  (4.23) $\Leftrightarrow W'(1-z^*)+0-U'(z^*) = \overline{U}'(z^*) = 0$
- $z_{t-1} \in [1-z^*, q]$: $z_t = 1-z_{t-1}$ and
  (4.23) $\Leftrightarrow W'(1-z_t)+\overline{U}'(1-z_{t-1})-\overline{U}'(z_t)=W'(1-z_t)+\overline{U}'(z_t)-\overline{U}'(z_t)=0$
- $z_{t-1} \in (q, 1]$: $z_t = 1-q$ and
  (4.23) $\Leftrightarrow W'(q)+\frac{1}{b}U'(q)-U'(1-q) = \frac{1}{b}U'(q)-\overline{U}'(1-q) = 0$

Finally, the equality $\mathcal{L}_{v_t} = \nu_t + \theta_t - \beta_t = 0$ holds trivially

We turn now to case $(\mathbf{S_R})$. Here, z_t evolves following the feedback law showed in Figure 4.1(b), becoming periodic in at most two steps. We recall here the proposed optimal trajectory

$$\text{(4.13)} \qquad \begin{cases} (z_0, p, 1-p, p, 1-p, \dots) & \text{when } z_0 \in [0, 1-p) \\ (z_0, 1-z_0, z_0, 1-z_0, \dots) & \text{when } z_0 \in [1-p, p] \\ (z_0, 1-z_0, p, 1-p, p, \dots) & \text{when } z_0 \in (p, q] \\ (z_0, 1-q, p, 1-p, p, \dots) & \text{when } z_0 \in (q, 1] \end{cases}$$

We observe that case $z_0 \in [1-p, p]$ has been already studied in Proposition 4.2.2. We have again $u_t = \min(z_t, q)$, as in $(\mathbf{S_I})$.

To see the optimality of these trajectories we propose the following ℓ^1_+-multipliers: $\mu = \nu = 0$, $\alpha_t = b^t\alpha(z_t)$ and $\theta_t = \beta_t = b^t\theta(z_t)$ where

$$\alpha(z) \quad = \quad \begin{cases} U'(z) & \text{if } z \in [0, 1-p) \\ \frac{Q'(z)-b^2W'(1-z)}{1-b^2} & \text{if } z \in [1-p, p] \\ Q'(z) & \text{if } z \in (p, q] \\ 0 & \text{if } z \in (q, 1] \end{cases}$$

$$\theta(z) \quad = \quad \begin{cases} 0 & \text{if } z \in [0, 1-p) \\ \frac{b}{1-b^2}[\overline{U}'(1-z) - b\overline{U}'(z)] & \text{if } z \in [1-p, p] \\ b\overline{U}'(1-z) & \text{if } z \in (p, q] \\ U'(q) & \text{if } z \in (q, 1] \end{cases}$$

Again, the functions $\alpha(z)$ and $\theta(z)$ are continuous in the whole interval $[0, 1]$.

Finally, notice that within the interval $[1 - p, p]$, the multipliers coincide with those of Proposition 4.2.2. Furthermore, the values outside this interval repeat those defined above for the $(\mathbf{S_I})$ case, with a different partition of interval $[0, 1]$. Hence, the proof is immediate following arguments already seen in this proof and that of Proposition 4.2.2. Finally, it is easy to see that the discounted total benefit given by the optimal trajectory is effectively the proposed value function, which proves the theorem. ■

4.4 Conclusions

We have analyzed a discrete time model for the optimal harvesting of a renewable resource, where the space occupancy is limited and resowing is not immediately allowed (assuming that the maturity age of the resource is reached after one time step).

Using two different techniques, based on dynamic programming and mathematical programming, we have characterized the optimality of two myopic strategies leading to constant or greedy periodic trajectories. This approach has led us to provide a method to make explicit the optimal trajectory for every possible initial state, and the optimal transient decisions to be kept.

If the delay before resowing was not compulsory, that is to say if we remove constraint (4.2), then the reserved land can be considered as allocated to another virtual species with a maturity age equal to one. Then, the problem is easily solved and an optimal trajectory converges to a new sustainable state $\tilde{z}$, given by $\tilde{z} \in \arg\max_{z\in[0,1]} \overline{U}(z)$ in only one time step, the optimal trajectory being constant afterwards. In this case, the sustainable state is the same as the one of our model when condition $(\mathbf{S_L})$ or $(\mathbf{S_I})$ holds. The qualitative behavior of both problems is the same, with convergence to the same state. However, under condition $(\mathbf{S_R})$, the situation is different. In this case, if reserving the land was not mandatory then the sustainable state would change: we would have $\tilde{z} > 1 - \tilde{z}$. Furthermore, in the problem without constraint (4.2), there might be convergence to a sustainable state with $\tilde{z} = 1$, that is to say the alternative use is not present at all. This represents a situation where the alternative use does not provide enough benefit to endure by itself.

Finally, considering an alternative use of the resource appears to be similar to an abstract problem with an additional virtual species, but is not exactly equivalent, as it is shown in this paper. The message, from a decision support point of view, is that a different anticipation scheme might be required, before using a myopic decision.

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ARBITRARY MATURITY AGE

The following material was not included in the submitted article. It concerns, the general case where the maturity age of the resource is arbitrary, a more appropriate model to be applied to forest management. The rest of the hypothesis remain unchanged.

In §4.5 we present the new model and the optimization problem associated. Then in §4.6 we study the optimality of the constant trajectory and the optimality of the greedy periodic trajectory. Finally, we study the asymptotic behavior of the system, achieving to characterize it completely in the particular case where the benefit function associated to the alternative use is linear.

4.5 The optimization problem

For each period $t \in I\!N$ we denote $x_t \geq 0$ the area occupied by the resource reaching maturity in year t, while $\bar{x}_t \geq 0$ represents the area with resource beyond maturity (older than n) and y_t represent the land allocated to the alternative use, i.e. land that could be allocated to the resource at the following step. An alternative representation of the system in terms of the age distribution at time t is provided by the *state* $\mathbb{X}_t = \big((x_{t+n-1}, \ldots, x_{t+1}, x_t, \bar{x}_t), y_t\big)$ that describes the areas occupied in year t by resource with ages $1, \ldots, n-1, n$, over n and the area assigned to the alternative use y_t. Of course $y_t = 1 - \bar{x}_t - \sum_{i=0}^{n-1} x_{t+i}$ at every step so the variable y_t is redundant, but keeping it makes the notation lighter and the calculations clearer[1].

At each step we need to decide how much resource $u_t > 0$ to harvest. Assuming that only mature resource can be harvested we must have $u_t \leq \bar{x}_t + x_t$, and the available resource that remains unharvested will be exactly the resource beyond maturity at the following step

$$\bar{x}_{t+1} = \bar{x}_t + x_t - u_t \tag{4.24}$$

The second decision of each step is what fraction v_t of the land occupied by the alternative use will be reallocated to the resource $0 \leq v_t \leq y_t$. From now on, we eliminate the variable v_t, given that $x_{t+n} = v_t$, i.e., the resource set in year t will reach maturity in year $t + n$

$$0 \leq x_{t+n} \leq y_t. \tag{4.25}$$

The total available area $u_t + y_t$ must be assigned either to the resource or to the alternative use

$$x_{t+n} + y_{t+1} = u_t + y_t. \tag{4.26}$$

[1]To retrieve the model of the first part, set $n = 1$ and $z_t = \bar{x}_t + x_t$ and notice that this implies $y_t = 1 - z_t$.

As before, the benefit obtained is $\sum_{t=0}^{\infty} b^t[U(u_t) + W(y_t)]$ where b, U and W fulfill Assumption A0.

Observe that the state evolution consists of an age-shift dynamics, except for $\bar{x}_t$, x_{t+n} and y_t which are controlled by the harvesting/sowing policy. Although we will not use these dynamics explicitly, the state $\mathbb{X}_t$ will be useful in describing the asymptotic behavior of the system. Notice that we do not control $\mathbb{X}_0$ which corresponds to the initial state reflecting the age-class composition of the system at time $t = 0$, so that, denoting $\boldsymbol{x}, \bar{\boldsymbol{x}}, \boldsymbol{u}, \boldsymbol{y}$ the infinite sequences of states and controls, the problem to be solved may be stated as

$$P(\mathbb{X}_0) \begin{cases} \text{maximize} \quad \sum_{t=0}^{\infty} b^t[U(u_t) + W(y_t)] \\ \text{subject to (4.24), (4.25) and (4.26)} \\ \boldsymbol{x}, \bar{\boldsymbol{x}}, \boldsymbol{y}, \boldsymbol{u} \in \ell_+^{\infty} \text{ with } \mathbb{X}_0 \text{ given.} \end{cases}$$

The existence of solutions for $P(\mathbb{X}_0)$ follows identically to that of problem $P(z_0)$ defined in (4.5).

We denote Δ the set of all initial states $\mathbb{X}_0 \in \mathbb{R}_+^{n+2}$ such that $\bar{x}_0 + \sum_{t=0}^{n-1} x_t + y_0 = 1$. Clearly enough the constraints (4.24) and (4.26) imply that $\mathbb{X}_t \in \Delta$ for all $t \in \mathbb{N}$. We also denote Δ^0 the set of states with $\bar{x}_0 = 0$, and we observe that an initial state $\mathbb{X}_0 \in \Delta$ yields the same optimal value and harvesting policy as $\tilde{\mathbb{X}}_0 \in \Delta^0$ where $\tilde{\mathbb{X}}_0 = \big((x_{n-1}, \dots, x_1, \bar{x}_0 + x_0, 0), y_0\big)$.

4.6 Stationary optimal trajectories

In this section we study the *greedy periodic policy* and the *constant policy*, characterizing the states for which they are optimal.

4.6.1 Greedy periodic policy

We recall the definition of a greedy periodic policy, setting it in the new context.

Definition 4.2.1. It consists in harvesting all the resource available and reallocating to the resource all the space that was assigned to the alternative use

$$u_t = \bar{x}_t + x_t, \qquad x_{t+n} = y_t.$$

This policy generates a cyclic trajectory of period $n+1$: $x_{t+n} = y_t$ and $y_{t+1} = x_t$ for all $t \geq 1$. We call such trajectories *greedy periodic cycles (GPC)*. An example of a GPC when $n = 2$ is

$$\begin{pmatrix}\bar{x}_0\\ x_0\\ x_1\end{pmatrix} y_0 \to \begin{pmatrix}0\\ x_1\\ y_0\end{pmatrix} \bar{x}_0{+}x_0 \to \begin{pmatrix}0\\ y_0\\ \bar{x}_0{+}x_0\end{pmatrix} x_1 \to \begin{pmatrix}0\\ \bar{x}_0{+}x_0\\ x_1\end{pmatrix} y_0 \to \begin{pmatrix}0\\ x_1\\ y_0\end{pmatrix} \bar{x}_0{+}x_0 \to \cdots$$

The benefit of the periodic trajectory generated by the greedy policy from any $\mathbb{X}_0 \in \Delta^0$ is

$$J_G(\mathbb{X}_0) = \tfrac{1}{1-b^{n+1}}\big[\textstyle\sum_{i=0}^{n-1} b^i U(x_i) + b^n U(y_0) + W(y_0) + \sum_{i=1}^{n} b^i W(x_{i-1})\big].$$

We recall that Δ^p is the set of states whose value function is $V(\mathbb{X}){=}J_G(\mathbb{X})$. It is characterized in the following proposition

Proposition 4.6.1. $\mathbb{X}_0 = \big((x_{n-1},\ldots,x_1,x_0,0),x_n\big) \in \Delta^p$ *if and only if*

(4.27) $$b^{n-1}[U'(x_t) + bW'(x_t)] \geq b^n U'(x_{t+1(n+1)}) + W'(x_{t+1(n+1)}) \textit{ for all } x_t > 0$$

where the notation $t(n)$ *denotes the integer* t *modulo* n.

Proof. To prove the necessity of condition (4.27) take $\mathbb{X}_0 \in \Delta^p$ so that the GPC is optimal for $P(\mathbb{X}_0)$. Suppose that $x_n > 0$ and consider the perturbed trajectory that consists in allocating $x_n - \epsilon$ to the resource at $t=0$ instead of x_n, keeping this extra land in the alternative use for one period and allocating $x_0{+}\epsilon$ at $t=1$, after which we continue with a GPC. If the original trajectory is optimal, the benefit obtained V with it must be greater or equal to the $\tilde{V}$ obtained with the new trajectory, this is

$$\begin{aligned}\tilde{V} - V \;=\; & b[W(x_0{+}\epsilon) - W(x_0)] + \tfrac{b^n}{1-b^{n+1}}[U(x_n{-}\epsilon) - U(x_n)] + \tfrac{b^{n+1}}{1-b^{n+1}}[U(x_0{+}\epsilon) - U(x_0)]\\ & + \tfrac{b^{n+1}}{1-b^{n+1}}[W(x_n{-}\epsilon) - W(x_n)] + \tfrac{b^{n+2}}{1-b^{n+1}}[W(x_0{+}\epsilon) - W(x_0)] \geq 0.\end{aligned}$$

Dividing by ϵ and letting it to 0 yields (4.27) for x_n. To prove it of all $x_j{>}0$ with $j{=}0,\ldots,n{-}1$ it suffices to repeat the reasoning perturbing the optimal trajectory at $t{=}j{+}1$ instead of $t{=}0$.

To prove the converse we claim first that condition (4.27) holds for all x_t even if $x_t = 0$.

We prove it by contradiction: if this was not the case, take x_i such that (4.27) holds (we know that such x_i exists because there is at least one $x_i > 0$) and suppose that (4.27) does not hold for $x_{i+1(n+1)}{=}0$.

$$\begin{aligned}b^{n-1}[U'(x_i){+}bW'(x_i)] \;&\geq\; b^n U'(0) + W'(0)\\ b^{n-1}[\,U'(0) + bW'(0)\,] \;&<\; b^n U'(x_{i+2(n+1)}) + W'(x_{i+2(n+1)})\end{aligned}$$

adding up we get the contradiction

$$b^{n-1}[U'(x_i) + bU'(x_{i+2})] + b^n W'(x_i) + W'(x_{i+2}) > b^{n-1}(1+b)U'(0) + (1+b^n)W'(0)$$

By induction we get (4.27) for all t.

Now, we prove the optimality of the stationary trajectory using the Karush-Kuhn-Tucker (KKT) theorem. Consider the Lagrangian

(4.28)
$$\begin{aligned}L &= \sum_{t=0}^{\infty} b^t\big[U(u_t) + W(y_t)\big] + \sum_{t=0}^{\infty}\mu_t u_t + \sum_{t=1}^{\infty} \bar{\lambda}_t \bar{x}_t + \sum_{t=n}^{\infty} \lambda_t x_t \\ &+ \sum_{t=0}^{\infty} \alpha_t(\bar{x}_t + x_t - u_t - \bar{x}_{t+1}) + \sum_{t=0}^{\infty} \theta_t(u_t + y_t - x_{t+n} - y_{t+1}) + \sum_{t=0}^{\infty} \nu_t(y_t - x_{t+n})\end{aligned}$$

together with the following ℓ^1-multipliers

$$\begin{cases} \mu_t = \lambda_t = 0 \\ \theta_t = \frac{b^{t+1}}{1-b^{n+1}}[b^n U'(x_t) + W'(x_t)] \\ \alpha_t = b^t U'(x_t) + \theta_t \\ \bar{\lambda}_t = \alpha_{t-1} - \alpha_t \\ \nu_t = \theta_{t-1} - \theta_t - b^t W'(x_{t-1}) \end{cases}$$

Only the non-negativity of $\bar{\lambda}$ and ν is not evident, and it is given by (4.27). The complementary slackness is easily seen and the stationarity of the Lagrangian (4.28) follows after some computation[2]. Hence the proposed trajectory is optimal. ■

4.6.2 Constant policy

We begin by recalling the definition of a sustainable state.

Definition 4.2.5. A state is called sustainable if it is invariant under the optimal policy.

The existence of such a state is not completely obvious. It has to be of the form $\mathbb{X}^* = \big((x, \dots, x, \bar{x}), y\big)$ with $y \geq x$ and its optimal policy should be the

Definition 4.2.4 (constant policy). It consists in harvesting and allocating always a constant quantity

$$u_t = x_{t+n} = x$$

so that the trajectory is constant: $\mathbb{X}_t = \big((x, x, \dots, x, \bar{x}), y\big)$

The benefit of the constant trajectory from the initial condition $\mathbb{X}_t = \big((x, x, \dots, x, \bar{x}), y\big)$ is given by

$$J_S(\mathbb{X}) = \frac{U(x)+W(y)}{1-b}.$$

[2]Observe that $x_{t+1} = x_{t-n}$ for all $t \geq n$.

Clearly, if the constant policy is to be optimal, the area of overmature trees has to be zero, $\bar{x}=0$. Otherwise, a policy that harvests $\bar{x}+x$ at $t=0$ and leaves $\bar{x}+y$ in the alternative use in all other periods would be more convenient, contradicting optimality. For the rest of this paper we denote $\sigma_i = b^i/(1-b^i)$. We claim that the sustainable states are exactly the points of the previous form that are solution to the following optimization problem

$$(S)\begin{cases} \text{maximize} & n\,\sigma_n U(x) + \sigma_1 W(y) \\ \text{subject to} & 0 \le x \le y \\ & nx + y = 1 \end{cases}$$

Our claim readily implies that there is always a sustainable state, and that it is unique if at least one of the benefit functions happens to be strictly concave.

Proposition 4.6.2. *The sustainable states are exactly the points of the form* $\mathbb{X}^* = \big((x^*,\ldots,x^*,0),y^*\big)$ *where the pair* (x^*,y^*) *is solution to the problem* (S).

Proof. To characterize the solutions to the problem (S), we consider the Lagrangian

$$L_S = n\,\sigma_n U(x) + \sigma_1 W(y) + p(1-nx-y) + q(y-x) + rx \tag{4.29}$$

The solution falls naturally into three mutually exclusive cases

$(\mathbf{S_L})$	If $\sigma_n U'(0) \le \sigma_1 W'(1)$ then $x^*=0$ and $y^*=1$	$p=\sigma_1 W'(1)$, $q=0$ and $r=n[\sigma_1 W'(1)-\sigma_n U'(0)]>0$.
$(\mathbf{S_I})$	If $\sigma_n U'(x^*) = \sigma_1 W'(y^*)$ with $y^*>x^*>0$ and $nx^*+y^*=1$	$p=\sigma_1 W'(y^*)$ and $q=r=0$.
$(\mathbf{S_R})$	If $\sigma_n U'(\frac{1}{n+1}) \ge \sigma_1 W'(\frac{1}{n+1})$ then $x^*=y^*=\frac{1}{n+1}$	$p=\frac{1}{n+1}[n\sigma_n U'(\frac{1}{n+1})+\sigma_1 W'(\frac{1}{n+1})]$ $q=\frac{n}{n+1}[\sigma_n U'(\frac{1}{n+1})-\sigma_1 W'(\frac{1}{n+1})]$ and $r=0$.

Let us show that the stationary trajectory $x_t=x^*$, $\bar{x}_t=0$, $y_t=y^*$, $u_t=x^*$, is optimal for $P(\mathbb{X}^*)$. To this end, we consider the Lagrangian (4.28) and the following set of ℓ^1_+-multipliers

$$\begin{cases} \mu_t = 0 \\ \theta_t = b^t\sigma_1 W'(y^*) + \frac{b^{t+1}(1-b^n)}{1-b^{n+1}}[\sigma_n U'(x^*)-\sigma_1 W'(y^*)]_+ \\ \alpha_t = b^t U'(x^*)+\theta_t \\ \bar{\lambda}_t = \alpha_{t-1}(1-b) \\ \lambda_t = \frac{b^t}{\sigma_n}[\sigma_1 W'(y^*)-\sigma_n U'(x^*)]_+ \\ \nu_t = \frac{b^t(1-b)(1-b^n)}{1-b^{n+1}}[\sigma_n U'(x^*)-\sigma_1 W'(y^*)]_+ \end{cases}$$

where $[x]_+$ stands for the positive part of x. A cumbersome verification shows that $\nabla L = 0$ and we have complementary slackness which implies that the proposed trajectory is a stationary point for $P(\mathbb{X}^*)$. Hence, it is an optimal solution and $\mathbb{X}^*$ is sustainable.

To prove the converse let $\mathbb{X}=\big((x,\ldots,x,0),y\big)$ be a sustainable state. Let us see that (x,y) belongs to the solution set of (S).

We claim that when $x>0$ then $\sigma_n U'(x) \geq \sigma_1 W'(y)$. Indeed, let us perturb the optimal harvesting policy as follows: at time $t=0$ we allocate $x-\epsilon$ to the resource and $y+\epsilon$ to the alternative use, instead of x and y. We keep this extra ϵ in the alternative use until stage $t=n$ when we have exactly $x-\epsilon$ area of resource reaching maturity. At this stage, we allocate x to the resource and we return to the stationary trajectory. The benefit derived from this perturbed policy must be less than the one obtained with the optimal one, which gives

$$\tilde{V}-V = b^n[U(x-\epsilon)-U(x)] + \textstyle\sum_{j=1}^n b^j[W(y+\epsilon)-W(y)] \leq 0.$$

Dividing by ϵ and letting $\epsilon \to 0$ we deduce $\sigma_n U'(x) \geq \sigma_1 W'(y)$ as claimed.

Moreover, if $y>x$ then we have $\sigma_n U'(x) \leq \sigma_1 W'(y)$. To prove it, consider the following alternative trajectory: at time $t=0$ allocate $x+\epsilon$ to the resource and $y-\epsilon$ to the alternative use instead of x and y respectively. Keep $y-\epsilon$ in the alternative use up to $t=n$, when there is $x+\epsilon$ of the resource reaching maturity. At this stage, allocate x and y again, returning to the stationary trajectory. The difference of benefit is

$$\tilde{V}-V = b^n[U(x+\epsilon)-U(x)] - \textstyle\sum_{j=1}^n b^j[W(y+\epsilon)-W(y)] \leq 0.$$

Dividing by ϵ and letting $\epsilon \to 0$ we deduce $\sigma_n U'(x) \leq \sigma_1 W'(y)$ as claimed.

Using these inequalities and the constrains $y \geq x$ and $nx+y=1$ it follows that (x,y) is a solution to (S). ■

Notice that when $(\mathbf{S_R})$ holds, the constant policy is a particular GPC and so $\mathbb{X}^* \in \Delta^p$. The following proposition summarizes the relationship between $\mathbb{X}^*$ and Δ^p.

Proposition 4.6.3. $(\mathbf{S_R})$ *holds* $\iff \mathbb{X}^* \in \Delta^p \iff \Delta^p \neq \phi$

Proof. Some of the implications are trivial: if $(\mathbf{S_R})$ holds, then $\mathbb{X}^* = \big((\frac{1}{n+1},\ldots,\frac{1}{n+1},0),\frac{1}{n+1}\big) \in \Delta^p$ which is non-empty. The proof is finished by showing that $[\Delta^p \neq \phi \Rightarrow (\mathbf{S_R})$ holds$]$. To this end, let $\mathbb{X} \in \Delta^p$ and chose x_t such that $x_t \geq \frac{1}{n+1}$ and $x_{t+1(n+1)} \leq \frac{1}{n+1}$. Then we have

$$\begin{aligned}
& b^{n-1}[U'(\tfrac{1}{n+1}) + bW'(\tfrac{1}{n+1})] \geq b^{n-1}[U'(x_t)+bW'(x_t)] \geq \\
& \quad \geq b^n U'(x_{t+1(n+1)}) + W'(x_{t+1(n+1)}) \geq b^n U'(\tfrac{1}{n+1}) + W'(\tfrac{1}{n+1}) \\
\Longrightarrow \quad & b^{n-1}(1-b)U'(\tfrac{1}{n+1}) \geq (1-b^n)W'(\tfrac{1}{n+1}) \\
\Longleftrightarrow \quad & \sigma_n U'(\tfrac{1}{n+1}) \geq \sigma_1 W'(\tfrac{1}{n+1})
\end{aligned}$$

which readily implies that $(\mathbf{S_R})$ holds. ■

4.7 Asymptotic behavior

We start by proving that after finitely many steps, any optimal trajectory becomes *greedy*.

Proposition 4.7.1. *Along the optimal trajectory* $\bar{x}_t = 0$ *for all* $t \geq 2n$.

Proof. The proof is very similar to that of [1, Proposition 4.1], we include it for the sake completeness. The proof is divided in two stages. First note that in each interval of length $n+1$ such as $p, \ldots, p+n$ there is at least one $\bar{x}_t = 0$. Indeed, if this was not the case then at time $p-1$ we could harvest a small additional area $\epsilon > 0$, leave it as alternative use land for one period, after which it is reallocated to the resource, i.e., we allocate $x_{p+n} + \epsilon$ instead of x_{p+n}. The overmature areas are modified as $\bar{x}_t - \epsilon$ for $t = p, \ldots, p+n$ and after $p+n$ we rejoin the original optimal strategy. This modified trajectory would increase the benefit by an amount $b^{p-1}[U(u_{p-1}+\epsilon) - U(u_{p-1})] + b^p[W(y_p+\epsilon) - W(y_p)] > 0$ contradicting optimality. In particular, in the interval of time $[n, 2n]$ there is at least one time T such that $\bar{x}_T = 0$.

Next observe that for $t \geq n$ we have $\bar{x}_t = 0 \Rightarrow \bar{x}_{t+1} = 0$. To see this we proceed again by contradiction: if $\bar{x}_t = 0 < \bar{x}_{t+1}$ then $u_t < \bar{x}_t + x_t = x_t$ which means that at stage t we do not harvest all the available resource. If we backtrack to stage $t-n$ when x_t was allocated, we could have taken out a small area $\epsilon > 0$ and leave it as alternative use land for another extra period making an additional benefit of $b^{t-n+1}[W(y_{t-n+1}+\epsilon) - W(y_{t-n+1})] > 0$, after which this ϵ is returned to the resource so that at stage $t+1$ the area reaching maturity $x_{t+1}+\epsilon$ compensate the loss of over-mature area $\bar{x}_{t+1} - \epsilon$. This trajectory allows to harvest the same areas as in the original strategy, except at stage $t-n+1$ where we make an extra benefit, which contradicts optimality.

Combining the previous properties we may conclude: there exists $T \in [n, 2n]$ such that $\bar{x}_t = 0$ for all $t \geq T$. ■

4.7.1 *U* strictly concave and *W* linear: Asymptotic convergence

We turn now to the special case where the benefit function of the resource U is strictly concave and that of the alternative use is linear, $W(y) = \alpha y$, $\alpha > 0$. We claim that the stationary trajectories studied in the previous section are typical, in the sense that any optimally managed system converges either to the sustainable state or to the set Δ^p.

In this particular case, the set Δ^p is simply expressed as (see Proposition 4.6.1)

$$\Delta^p = \left\{ \mathbb{X} \in I\!R_+ \,/\, \sigma_n[U'(x_t) - bU'(x_{t+1(n+1)})] \geq b\alpha \quad \forall t = 0, \ldots, n \right\} \tag{4.30}$$

and it is non empty iff $(\mathbf{S_R})$ holds, i.e., whenever $\sigma_n U'(\frac{1}{n+1}) \geq \sigma_1 \alpha$.

To characterize the asymptotic behavior, we use the linearity of W to write our problem as a classical two species forest harvesting problem, in order to use results of [1]. Due to the proposition above we restrict ourselves to greedy trajectories.

As $y_t \geq x_{t+n}$, we can always write $y_t = x_{t+n} + w_t$ with $w_t \geq 0$. Our problem is now expressed as

$$\tilde{P}(\mathbb{X}_0) \begin{cases} \text{maximize} & \sum_{t=0}^{\infty} b^t [U(x_t) + \alpha(x_{t+n} + w_t)] \\ \text{subject to} & x_t + w_t = x_{t+n+1} + w_{t+1} \\ & \boldsymbol{x}, \boldsymbol{w} \in \ell_+^\infty, \quad \text{with } \mathbb{X}_0 \text{ given.} \end{cases}$$

and the objective function can be written as

$$\sum_{t=0}^{\infty} b^t [U(x_t) + \tfrac{\alpha}{b^n} x_t + \alpha w_t] - \sum_{t=0}^{n-1} b^t \tfrac{\alpha}{b^n} x_t$$

where the last term is given by the initial condition. Hence, $\tilde{P}(\mathbb{X}_0)$ can be seen as a two species problem where species $\boldsymbol{x}$ has maturity age $n+1$ and benefit function $\tilde{U}(x_t) = U(x_t) + \frac{\alpha}{b^n} x_t$ and $\boldsymbol{w}$ is annual with linear benefit function $W(w) = \alpha w$.

It is stated in [1] that if $\mathbb{X}_0$ is such that its optimal trajectory becomes greedy in finite time, then this trajectory converges to $\tilde{\Delta}^p$ the set of GPCs. The result holds when one of the benefit functions is concave non-decreasing, provided that the other is strictly concave (see [1, Theorem 4.6] and the following commentary). Even more, the set $\tilde{\Delta}^p$ comprises only the sustainable state $\tilde{\mathbb{X}}$ whenever the annual species is different from zero at the sustainable state (see [1, Section 3]). The unique sustainable state can have one of the three following structures:

$$\begin{array}{rll} (i) & \tilde{\mathbb{X}} = \big((0,0,\dots,0,0),1\big) & \text{if } \sigma_{n+1}\tilde{U}'(0) \leq \sigma_1 \alpha \\ (ii) & \tilde{\mathbb{X}} = \big((\tilde{x},\tilde{x},\dots,\tilde{x},0),\tilde{w}\big) & \text{if } \exists\, \tilde{x} \in (0,\frac{1}{n+1}) \text{ such that } \sigma_{n+1}\tilde{U}'(\tilde{x}) = \sigma_1 \alpha \\ (iii) & \tilde{\mathbb{X}} = \big((\frac{1}{n+1},\frac{1}{n+1},\dots,\frac{1}{n+1},0),0\big) & \text{if } \sigma_{n+1}\tilde{U}'(\frac{1}{n+1}) \geq \sigma_1 \alpha \end{array}$$

Hence, $\tilde{\Delta}^p = \{\tilde{\mathbb{X}}\}$ in cases (i) and (ii). And using Theorem 2.3.4, it is easy to characterize it in (iii): $\mathbb{X}_0 \in \Delta^p$ iff for all $t = 0, 1, \dots, n$ we have:

$$\sigma_{n+1}\big[\tfrac{1}{b}\,\tilde{U}'(x_t) - \tilde{U}'(x_{t+1(n+1)})\big] \geq \alpha \tag{4.31}$$

It is easy to see that (i) holds iff we are in case $(\mathbf{S_L})$:

$$\sigma_{n+1}\tilde{U}'(0) \leq \sigma_1 \alpha \iff \sigma_{n+1}[U'(0) + \tfrac{\alpha}{b^n}] \leq \sigma_1 \alpha \iff \sigma_n U'(0) \leq \sigma_1 W'(1).$$

The equivalences $(ii) \iff (\mathbf{S_I})$ and $(iii) \iff (\mathbf{S_R})$ follow analogously. The sustainable states of both systems are directly identifiable,

$$\begin{array}{rlcll} (i) & \tilde{\mathbb{X}} = \big((0,0,\ldots,0,0),1\big) & \iff & (\mathbf{S_L}) & \mathbb{X}^* = \big((0,0,\ldots,0),1\big) \\ (ii) & \tilde{\mathbb{X}} = \big((\tilde{x},\tilde{x},\ldots,\tilde{x},0),\tilde{w}\big) & \iff & (\mathbf{S_I}) & \mathbb{X}^* = \big((x^*,\ldots,x^*,0),x^*+\tilde{w}\big) \\ (iii) & \tilde{\mathbb{X}} = \big((\frac{1}{n+1},\frac{1}{n+1},\ldots,\frac{1}{n+1},0),0\big) & \iff & (\mathbf{S_R}) & \mathbb{X}^* = \big((\frac{1}{n+1},\ldots,\frac{1}{n+1},0),\frac{1}{n+1}\big) \end{array}$$

as well as the GPC's sets,

$$\begin{aligned} & \tilde{\mathbb{X}}_0 = \big((x_n, x_{n-1},\ldots,x_1,x_0,0),0\big) \in \tilde{\Delta}^p \\ \iff & \sigma_{n+1}\big[\tfrac{1}{b}U'(x_t) + \tfrac{\alpha}{b^{n+1}} - U'(x_{t+1(n+1)}) - \tfrac{\alpha}{b^n}\big] \ge \alpha && \text{for all } t = 0,\ldots,n \\ \iff & b^n\big[\tfrac{1}{b}U'(x_t) - U'(x_{t+1(n+1)})\big] \ge \alpha(1-b^n) && \text{for all } t = 0,\ldots,n \\ \iff & b^{n-1}U'(x_t) + b^n W'(x_t) \ge b^n U'(x_{t+1(n+1)}) + W'(x_{t+1(n+1)}) && \text{for all } t = 0,\ldots,n \\ \iff & \mathbb{X}_0 = \big((x_{n-1},\ldots,x_1,x_0,0),x_n\big) \in \Delta^p \end{aligned}$$

Theorem 4.6 in [1] gives immediately that every optimal trajectory of $\tilde{P}(\mathbb{X}_0)$ converges to a GPC in the sense that

$$\lim_{t\to\infty} \text{dist}(\mathbb{X}_t, \tilde{\Delta}^p) = 0.$$

and in particular in cases (i) or (ii), the system converges to the sustainable state: $\lim_{t\to\infty} \mathbb{X}_t = \tilde{\mathbb{X}}$. Hence, we can conclude that every optimal trajectory of $P(\mathbb{X}_0)$ converges to a GPC in the sense that

$$\lim_{t\to\infty} \text{dist}(\mathbb{X}_t, \Delta^p) = 0.$$

and in particular if $(\mathbf{S_L})$ or $(\mathbf{S_I})$ hold, the system converges to the sustainable state: $\lim_{t\to\infty} \mathbb{X}_t = \mathbb{X}^*$.

4.8 Conclusions

In this last part we have included delay of maturity in the model, in order to make it suitable to represent a forest stand. We have seen that the optimal trajectory becomes greedy after finitely many steps. The main result of this second part is the characterization of the asymptotic behavior in the case of a linear benefit function. In this case, we can write our problem as in [1], getting rid of constraint (4.2) by adding an extra age class to the resource. We get that whenever $y^* > x^*$ the system converges to the sustainable state. If on the contrary, $y^* = x^*$ the system converges asymptotically to Δ^p.

Part II

Other characteristics of the forest growth process

Chapter 5

A bibliographical review

The literature concerning forests models is very extended, we do not attempt to make a complete review here. Even a survey comprising only stand-level, distance independent, mixed forest models is out of the scope due to its immensity. Forest growth models have been developed with various levels of detail, and with an emphasis on either mechanistic process representation or on accurate long-term forecasting. The most appropriate model types depend on intended use, stand characteristics, and other circumstances. We focus on the so-called “empirical" models, i.e., models derived from large amounts of field data that describe growth rate as a regression function of variables such as site index, age, tree density, and basal area. In this chapter, we present a forest growth model classification to help situating the models we will treat afterwards with respect to the universe of forest growth models. Then, we describe five articles,where population dynamics models for mixed forests are designed and some management issues are discussed as examples.

5.1 Forest growth models classification

There is a huge amount of forest growth models and there is no consensus of the forestry community about a systematic classification of them. We follow Porté and Bartelink review [28] to classify the models and present them in a clear order. It is worth mentioning that there exist several different, often contradictory classifications. We have chosen Porté and Bartelink among other authors because their main purpose is to classify models designed for mixed forest growth and because, in our opinion, their three-levels, binary classification tree is unambiguous, universal, easy to understand and to use.

The first classification criteria used by the authors is the smallest unit identified distinguishing in the first place between *tree-level* and *stand-level* models. Tree-level models, also called individual models, describe and keep track of each individual tree on the stand. They generally specify the diameter, and possibly height and/or other variables. Stand-level models consider the forest as a unit and describe the stand by a small number of aggregate “macro” variables, such as mean diameter, top height, trees per hectare, etc.

As a second criteria spatial dependence is considered, i.e., both tree-level and stand-level models can be divided into distance dependent and distance independent models. And the third criterion describes whether or not forest heterogeneity is taken into account. In summary, a sketch of the classification is

1. Stand-level	→	(a) Distance dependent		
	→	(b) Distance independent	→	i. Average tree models
			→	ii. Distribution models
2. Tree-level	→	(a) Distance dependent		
	→	(b) Distance independent	→	i. Gap models
			→	ii. Non gap models

1(a) Stand-level, distance dependent

Here the forest is described as a mosaic of patches, each of them characterized by its position in the forest. Each patch, has its own dynamics and interacts with its neighbors. Inside the patch the forest is described by a distance independent model.

1(b) Stand-level, distance independent

The forest is considered as a unit, without any spatial organization. We can distinguish the models that take the heterogeneity into consideration (distribution models) and those that do not (average tree models).

1(b)i. Average tree models With a history of over 250 years, yield tables are the oldest models in forestry science. The quantities taken into account were most commonly stand basal area and volume. Not as often the number of trees, the average diameter and the height increments were also included. For mixed forest, the tables provide output values for each species. For many purposes this type of model is still adequate, but they cannot deal easily with arbitrary deviations from the average curve. Sometimes more flexibility is needed and dynamic modelling is desirable, hence the model is represented by a system of difference or differential equations which allows to estimate the variation of the modelled stand characteristics, as a function of the time elapsed and the initial conditions. Only a few models describe the forest with the interacting species, i.e., linking the evolution of a species with the attributes of other species. Jogiste [14] (see Subsection 5.3) and Puetmann et al [29] introduced the effect of species proportion on the basal area increment per species.

1(b)ii. Distribution models Here, the average and total dimensions of the stand per tree species are modeled, the difference with the previous models being the partial consideration of the natural variability of the trees in a stand. Each of the chosen characteristics is described by a distribution function, either continuous or discrete. When we talk of discrete distribution functions is where the *classes* appear, i.e., trees are divided into classes of equivalence according to their age, height, diameter or any other attribute. It is worth mentioning that inside each class trees are considered identical.

2(a) Tree-level, distance dependent

These are the most detailed models, where each tree is independently considered and interacts with its neighbors. They may require a lot of computation time, but they allow the simulation of inter-tree competition and a detailed prediction of the tree attributes. For example, a ratio of influence can be defined and every tree will affect and be affected by any other tree within it. These models are truly diverse in the modelling approaches, and we are not discussing them any further.

2(b) Tree-level, distance independent

Generally these models are lists of trees where the characteristics of each tree are known but its position is not considered.

2(b)i. Gap models Here, the forest is simulated as a group of patches or gaps, each of them described by a list of individual trees. Inside the gap the dynamics are described at the individual tree level. Most of gap models are mainly used to understand the forest successional patterns and processes in a canopy gap area created by tree falling. Therefore, it often matches the crown size of a dominant tree.

2(b)ii. Non gap models There are few examples of distance independent, non gap models. Individual tree growth is modelled as a function of the tree values and the average values of the stand.

Tree-level, high-dimensional models, especially spatial ones, are attractive for their ability to represent biological knowledge in an easier, more natural way. Simulation of a large variety of hypothetical situations is possible. On the contrary, aggregated models summarize and make development patterns comprehensible and they make it easier to go beyond simulations and find general laws of behavior. From the decision-making point of view, incomplete information about a high-dimensional initial state limits forecast reliability, in addition to being a source of redundancy and over-parameterization, affecting the statistical precision of estimates.

Models that include biological processes, inter and intra species competition, and that could be used by forest managers would be very valuable but are still scarce. Today, sometimes the management decisions are taken by simulating a finite number of possible exploitation policies and comparing their results, imposing some long run goal such as a predetermined structure of the forest or maximum sustained forest yields.

We focus on the so-called “empirical" models, i.e., models derived from large amounts of field data that describe growth rate as a regression function of variables such as site index, age, tree density, and basal area. The key to the success of these models is the choice of independent variables to describe the growth. The empirical models are defined as opposites to the “process" models, that describe or simulate the dependence of growth on a number of very detailed interaction processes such as photosynthesis, respiration, decomposition and nutrient cycling. For example, Peng [27] writes “The major strength of the empirical approach is in describing the best relationship between the measured data and the growth-determining variables using a specified mathematical function or curve. They are also easily incorporated into diversified management analysis and silvicultural treatments, and are able to achieve greater efficiency and accuracy in providing quantitative information for forest management".

We present five articles, where population dynamics models for mixed forests are designed and some management issues are discussed.

5.2 J. Buongiorno, J.L. Peyron, F. Houllier, M. Bruciamacchie. Growth and management of mixed-species, uneven-aged forests in the French Jura: implications for economic returns and tree diversity [6]

In this paper the authors design a stand-level, distance independent, discrete time, nonlinear matrix growth model that was calibrated and validated with data from forests of the Jura mountains of France. The objective was to provide analytical tools to forests managers, wishing to compare some predefined criteria, like for example economic efficiency or ecological diversity, over a set of predefined policies.

The forests of the French Jura are particular because the traditional silviculture used is technically difficult: it requires the marking of the trees to be harvested by a skilled forester. But it is economically rewarding because it relies on natural regeneration and it keeps a natural look of the forest with a mix of firs, spruces and beeches of various size in the same area.

We have chosen to include this paper in our review because it treats a discrete time, distribution model and develops tools to guide the management. We point out that four different policies are defined and compared, but there is no attempt to characterize the optimum among a broader set of the feasible policies.

Nonlinear growth model

A matrix model is developed, with coefficients that depend on the stand basal area (total area of tree stems measured at human breast height). No fauna or flora out of firs, spruces and beeches is recognized.

The stand state at time t is represented by the vector $\boldsymbol{y}_t = [y_{ijt}]$ where y_{ijt} represent the number of trees per unit area that are alive before the harvest, where i stands for the species i ($i=1,2,3$) and j for the diameter size class ($j=1,\ldots,n$). The control variable is the harvest at time t, represented by $\boldsymbol{h}_t=[h_{ijt}]$, and the dynamics are

(5.1)

$$\begin{cases} y_{i1(t+1)} = (1-b_{i1t}-m_{i1t})(y_{i1t}-h_{i1t}) + I_{it} \\ y_{ij(t+1)} = (1-b_{ijt}-m_{ijt})(y_{ijt}-h_{ijt}) + b_{i(j-1)t}(y_{i(j-1)t}-h_{i(j-1)t}) \qquad j=2,\ldots,n-1 \\ y_{in(t+1)} = (1-m_{int})(y_{int}-h_{int}) + b_{i(n-1)t}(y_{i(n-1)t}-h_{i(n-1)t}) \end{cases}$$

where

b_{ijt} fraction of trees of species i that stay alive and move from size class $j-1$ to j during the time interval from t to $t{+}1$. It is not constant, instead it is proposed to be a linear function of the total basal area and the diameter of the average tree in size-class j. It is expected to have a negative variation with the basal area, reflecting a slower growth rate at higher stand density.

m_{ijt} fraction of trees of species i and size j that die between t and $t+1$. Also considered a linear function of the total basal area and the diameter of the average tree in size-class j, it is expected to vary positively with the latter since larger and taller trees are more prone to windfall.

I_{it} ingrowth, i.e., the number of trees that will enter the smallest size class. It is an affine function of the stand density and the proportion of trees of the same species. It is expected to vary negatively with the former but positively with the latter. It also has a positive constant term, meaning that some ingrowth may occur, independently of stand state, due to seed dispersal from surrounding stands[1]

Model calibration and validation

All the parameters involved were estimated with data from two inventories, made 20 years apart, of 44 stands of the French Jura mountains that were approximately 8 hectares of land area each. These inventories listed the number of trees in each stand by species and by size, measured by tree diameter at breast height. There were nine size-classes, each one corresponding to a range of 5 cm. The number of trees harvested and dead (mainly by windfall) in each species-size class between the two inventories was known. The time unit was set at 5 years, which is approximately the current cutting cycle. The state of each stand at every time stage between the two inventories was estimated by interpolation.

The coefficients were estimated and their statistical significance at the 5% level was determined with different success. The authors affirm that the model badly estimates the ingrowth. The transitions of trees between size-classes is well estimated but, it resulted almost independent of basal area and diameter for beeches. And finally, mortality estimation was independent of the species which allowed the construction of a simplified model.

After the calibration and as way of validating the model, it was used to forecast the state of the stand at the time of the second inventory, taking the data of the first as the initial condition, for each of the 44 stands. The average over the 44 stands of the predicted values, was in general within that 95% confidence interval of the observed average. We observe that the same data is used to calibrate and validate the model.

Management criteria and policies

Two criteria are used to compare different management regimes:

[1] Today, there is a tendency to rely on natural regeneration instead of artificial sowing, due to the elevated costs of this activity. It is worth mentioning though, that according to Porté and Bartelink [28] the ingrowth is very difficult to model accurately, the best numerical simulations having errors of 30%.

Ecological criteria Two values are used to estimate the ecological biodiversity. One is simply the minimum number of trees in any species-size class. The other is the Shannon entropy of the distribution p_{ij}, where p_{ij} is the proportion of trees of species i and size-class j. The entropy is obviously maximized when trees are evenly distributed between size and species.

Economic criteria The economic criteria used are the present value of future revenues net of costs and the rate of return on the capital invested in the growing stock.

The authors evaluate all these criteria when the forest remains invariant. Hence, they start by finding the invariant states under each of the four following policies

(i) remove only the dead trees,

(ii) maintain the forest in a predefined state (in this case, the recorded initial state),

(iii) achieve economic efficiency, subject to a desired level of diversity,

(iv) maximize diversity, with or without constraints on economic returns.

For each policy, the steady state is numerically found and the criteria evaluated. In (iii) and (iv) the value of the basal area (B) is predetermined to avoid dealing with the nonlinearities that appear in the model (5.1). Trying to circumvent this suboptimality, the authors solve the problem with different values of B and chose the most convenient.

Concluding remarks

This study has resulted in a growth model for uneven-aged forest stands in the fir-spruce-beech forest of the Jura mountains in France. The data base which supports this model was sufficiently detailed, and there was enough variability in stand conditions, to arrive at satisfactory statistical estimates of the parameters.

The model describes average stand growth in the region of interest. Therefore, individual stand characteristics are taken into account by the model only insofar as they are reflected in the current stand state, defined by the number of trees in each species and size category. The model projections assume that silvicultural practices that have been done in the past, would continue in the future. Within these limits, the model appears to predict adequately average stand growth over 20 yr.

The decision to use a nonlinear growth model was justified by the strong inverse relationship that was apparent between the rate of diameter growth and stand basal area. But this nonlinearity of the model complicates considerably the study of management alternatives. This is why the authors keep the length of the cutting cycle constant, equal to five years.

The diversity and economic comparisons of this study deal mostly with steady states and do not take into account the initial condition of a specific stand, and the path followed from the initial condition to the steady state.

Within these limits, the results of this study document the fact that in uneven aged stands, very different steady states are possible, each leading to varied economic performances and forest stand diversities. When the object is tree diversity, it should be kept in mind that this diversity has a cost that may be at least approximated by comparing the income generated with what it would be without diversity constraints.

5.3 Kalev Jögiste. Productivity of mixed stands of spruce and birch affected by population dynamics: a model analysis [14]

A theoretical average-tree, discrete-time model is developed based on principles of dynamic programming and calibrated with data from mixed Norway spruce and birch forests. Of course, understanding the evolution of the bolewood production provides a basis for economic decision making.

Trees were divided into separate populations according to species and canopy class. Each stand comprises three populations: birch, overstory (dominant) spruce and suppressed spruce. Birch is not classified according to canopy class because, due to its shade intolerance, it occurs mainly in the upper layer. The populations are treated separately in the stand, and diameter growth is simulated separately as well.

An important aspect is the choice of independent variables to describe and predict the growth rate. In this work, both tree age and size have been included in the growth model as predictive variables. The author chooses the diameter of the tree as the growth measure, and its evolution is expressed as

$$D_p(t+1) = D_p(t) + f[D_p(t), G_B(t), G(t), t]$$

where the subscript p denotes the species and

$\mathrm{D_p}$: diameter at breast height of the mean tree of the population p

$\mathrm{G_B}$: birch proportion of the stand

G : stand basal area

Data and analysis

In particular the author considers the following regression equation

$$f[D_p, G_B, G, A_p] = a_0 + a_1 \ln(A_p) + a_2 D_p + a_3 G_B + a_4 G \tag{5.2}$$

(where A_p is the age of the mean tree of the population p) to be fitted separately for birch and the two spruce populations.

The model described in (5.2) needs empirical regressions of diameter growth on tree and stand variables. The necessary field measurements were collected in forest growth sites in Estonian forest growth conditions. Only trees with a diameter at breast height greater than 6 cm. were measured. The data recorded were: species, diameter and spruces' crown class as dominant or suppressed.

The stem number of the three populations (N_p) was recorded in each stand. Only the height and age of some trees was measured. To chose them, the population was divided in 4-cm diameter classes and 20% of each class was selected randomly. Sample trees were cored to measure radial increments and age. Their height was also recorded and the stand age (SA) was computed.

Basal area of populations are obtained as $G_p = \frac{\pi}{4} N_p D_p^2$, and the total stand basal area is obviously $G = G_B + G_{S_I} + G_{S_{II}}$, where B stands for birch population, and S_I and S_{II} for dominant and suppressed spruce populations.

Two separate regression equations were fitted to stem numbers as a function of the stand age.

$$\ln(N_B + N_{S_I}) = b_1 - b_2 \ln(SA) \qquad \text{(overstory stem number)}$$
$$\ln(N_{S_{II}}) = c_1 - c_2 \ln(SA)$$

Besides, decline in the birch proportion in the overstory stem number was modelled by a linear time function $\frac{N_B}{N_B + N_{S_I}} = \left(\frac{N_B}{N_B + N_{S_I}}\right)\Big|_{t=0} - \nu t$.

Tree height for volume calculations was predicted by a separate regression fit to data obtained during the field study:

$$H_p = d_0 \left(\frac{D_p}{D_p + 1}\right)^{d_1}$$

where H_p is the mean tree height.

Population volume was calculated by the formula of Laasasenaho (1982) [15]

$$V_p = \beta_0 N_p D_p^{\beta_1} H_p^{\beta_2} (H_p - 1.3)^{\beta_3}.$$

Results

Many simulations were performed with the adjusted model. The initial states where chosen in accordance with collected study material.

Since birch has a greater diameter growth than spruce, stand productivity should increase when birch proportion is greater. However, this conclusion is true only in middle-age stands because more birch in a stand inhibits the diameter growth of both species, due to the closure of the canopy. These two components (greater birch increment and growth inhibition by birch) is quantified in this article and the dependence of growth patterns on these factors is investigated.

From the simulation' results presented, it is apparent that at the beginning of the simulation age, the stands with birch dominance are superior to spruce stands both in terms of standing and total yield. But later, the spruce forests exceed the growth of stands with birch dominance.

The hypothetical birch population decrease coincides with observations from forestry practice, and many authors recommend cutting birch before spruce maturity because of the lower maturity age of birch.

Concluding remarks

The modelling attempt in this study presents a new approach to mixed-species stand dynamics. In the present case, the main tree growth is calculated for populations of different species and canopy classes.

The main result of this paper is the characterization of the bolewood accumulation pattern and its dependence on birch population. Silvicultural practice should consider this and intermediate cuttings are recommended in mixed stands with a high proportion of birch, to avoid the competition effect and the resultant decrease in diameter growth and stand productivity.

5.4 A.-S. Crépin. Multiple Species Boreal Forests - What Faustmann Missed

In this article the author analysis a continuous time model comprising three species that interact between themselves. There is no distance considered, no classification inside the species. Specifically, the model aims to represent dynamics in boreal forest ecosystems. Recent research in natural sciences shows that these dynamics are much more complex than what many models traditionally used in forestry economics reflect. The article shows that the optimal harvesting strategy for forest owners is history dependent and for some states of the forest more than one strategy may be optimal. This kind of phenomena are valid for a wide range of ecosystems that cover large surfaces and they do not depend on the choice of some specific function to model the non-linearity. This paper derives some optimal management rules to guide forest owners or decision makers who harvest several species. The harvesting rules are time-continuous.

Three-species boreal-forest model

The model used describes only the dynamics of conifers (spruce, pine), caduceus trees (aspen, birch) and large browsers (moose, elk). The general model is

$$\left\{\begin{array}{lclr} \dot{x} & = & G^x(x,y,z) & \text{herbivores} \\ \dot{y} & = & G^y(x,y,z) & \text{caduceus trees} \\ \dot{z} & = & G^z(x,y,z) & \text{conifers} \end{array}\right. \tag{5.3}$$

where x, y and z could be vectors representing categories of the population, like for example size or age classes or location in the territory. But this papers focuses on the effects of species' interactions and so the model used does not account for age and space. Hence x, y and z are scalars.

An example of specific model is presented and used in the numerical simulations

$$\left\{\begin{array}{lcl} G^x(x,y,z) & = & x - x^2 + a_{xy}xy + a_{xz}xz \\ G^y(x,y,z) & = & r_y y - y^2 - a_{yx}xy + a_{yz}z \\ G^z(x,y,z) & = & r_z z^2 - z^3 - a_{zx}xz - a_{zy}yz \end{array}\right. \tag{5.4}$$

Observe that pine exhibits a convex-concave growth, which is modelled using the function $z^2(r_z - z)$. When pine biomass is small, growth is convex because the term $r_z z^2$ dominates. When the population becomes larger, competition arises and growth becomes concave because the term z^3 dominates. This means that we depart a little from the traditional concavity assumption.

The author proves that depending on the value of the parameters of (5.4), there might be up to 15 feasible steady states (with 0, 1, 2 or 3 species present) and classifies them into stable, saddle or unstable. The phase space is divided in basins of attractions, so that trajectories that start anywhere in a region will end up in the same stable steady state.

General management rules

The author states the forest owner's optimization problem where the maximizing-welfare harvesting rule, $h^*(t) = (h_i^*(t))_{i\in\{x,y,z\}}$, is looked for. She considers an infinite horizon discounted problem with additive and concave profits.

$$
(P)\begin{cases}
\max_h & \int_0^{+\infty} e^{-\rho t}\big[\Omega_x(h_x)+\Omega_y(h_y)+\Omega_z(h_z)+\Omega_e(x,y,z)\big] \\
\text{s.t.} & \dot{x}=G^x(x,y,z)-h_x \\
 & \dot{y}=G^y(x,y,z)-h_y \\
 & \dot{z}=G^z(x,y,z)-h_z \\
 & x\geq 0, y\geq 0 \text{ and } z\geq 0
\end{cases}
$$

where $\Omega_i(h_i)$ represents profits from harvesting species $i \in \{x,y,z\}$ and Ω_e accounts for net benefits from environmental and recreational services.

Even though, the problem is not explicitly solvable in general (nor is it solvable with the specific dynamics proposed in (5.4)), some necessary conditions can be deduced. To this end, we define the current value Hamiltonian and the Lagrange function:

$$
\begin{aligned}
\mathcal{H}(x,y,z,h,\lambda,t) &= \textstyle\sum_i \big[\Omega_i(h_i)+\lambda_i(G^i(x,y,z)-h_i)\big]+\Omega_e(x,y,z) \\
\mathcal{L}(x,y,z,h,\lambda,\mu,t) &= \mathcal{H}(x,y,z,h,\lambda,t)+\textstyle\sum_i \mu_i(G^i(x,y,z)-h_i)
\end{aligned}
$$

Assuming existence of a solution $h^*(t)$ and multipliers ($\lambda_i(t)$ and $\mu_i(t)$), the necessary conditions for $h^*(t)$ to be optimal are:

1. $h^*(t)$ maximizes $\mathcal{H}$ subject to the constraints $G^i(x(t),y(t),z(t))-h_i(t) \geq 0$ for all $i \in \{x,y,z\}$ such that $i(t)=0$.
2. $\mu_i(t)$ are such that for all i, $\frac{\partial\mathcal{L}}{\partial h_i}=0$ for $(x,y,z)=(x(t),y(t),z(t))$, $h=h^*(t)$, $\lambda=\lambda(t)$ and $\mu_i(t)i(t)=0$, $\mu_i(t)(G^i(x,y,z)-h_i)=0$
3. the motion equations of the exploited ecosystem: $\frac{di}{dt}=G^i(x,y,z)-h_i$ for all $i \in \{x,y,z\}$
4. the necessary conditions for optimal harvest for all $i \in \{x,y,z\}$

   $$\frac{\partial\Omega_i(h_i^*)}{\partial h_i}-\lambda_i-\mu_i=0 \text{ or } h_i^*=0 \tag{5.5}$$

5. the shadow price equations for each species, for all $i,j \in \{x,y,z\}$

   $$\dot{\lambda}_i=\rho\lambda_i-\frac{\partial\Omega_e(x,y,z)}{\partial j}-\textstyle\sum_i(\lambda_i+\mu_i)\frac{\partial G^i(x,y,z)}{\partial j} \tag{5.6}$$

6. and the non-negativity, for all $i,j \in \{x,y,z\}$

   $$\mu_i(t)i(t)\geq 0 \text{ and } \mu_i(t)(G^i(x,y,z)-h_i)\geq 0.$$

Proposition 1 follows directly from (5.5)

Proposition 1 The optimal size for each species' harvest is such that the marginal value from harvesting more of the species equals the marginal value of retaining more of it in the ecosystem.

And if system (5.5) has a solution, then Proposition 2 follows from (5.6)

Proposition 2 In a steady state, the interest on a species' marginal value in the ecosystem equals the species' marginal environmental benefit plus the species' marginal benefit in maintaining its own and other species' stock.

From then on, the author considers only optimally managed systems where no species ever becomes extinct, discarding the non-negativity condition on the state variables. If the solution has at least one steady state, the author shows that the eigenvalues of such steady state come in pairs $\alpha, \rho - \alpha$. Proposition 3 follows directly

Proposition 3 Suppose $\rho > 0$. If the solution has a steady state, it is either unstable or saddle.

If the system (5.3) has several steady states, then for each of them, there is a value for ρ, say $\tilde{\rho}$, under which the steady state exhibits a local saddle-path property or has eigenvalues equal to zero and above which the steady state is locally unstable. This produces a series $(\tilde{\rho})$ of threshold values for ρ. Corollary 4 follows directly.

Corollary 4 Suppose the system (5.3) has several steady states. Let $\underline{\rho} = \min \tilde{\rho}$ and $\overline{\rho} = \max \tilde{\rho}$. If $\rho < \underline{\rho}$, there is a local saddle path that leads towards each steady state. If $\rho > \overline{\rho}$, all of the steady states are locally unstable. If $\underline{\rho} < \rho < \overline{\rho}$, some steady states are locally unstable while others have a local saddle path.

Concluding Remarks

This paper shows that a boreal forest, managed to maximize benefits derived from it, may have several optimal interior steady states. The path that is optimal to follow and thus the optimal steady state towards which the system converges, depend on the initial state of the system. It is worth mentioning that generally finding the steady states is a difficult task. Even when parameters are replaced with numerical values in (5.4), they cannot be analytically computed. They must be evaluated numerically.

What do these results imply for a manager? The existence of multiple steady states reveals the exploited ecosystem's dependency on history. What is optimal for one state of the system is not necessarily optimal for another. If the system had a unique interior steady state, the forest owner could reach the optimal saddle path by harvesting just enough for marginal costs to equal marginal benefits from harvest - including costs and benefits due to environmental changes and effects on other species (Proposition 1).

Here, marginal analysis is usually not enough to determine the optimal trajectory at given initial points. One needs more information, the optimal solution depends on the initial state. Furthermore, small management mistakes could lead to a different basin of attraction and the future harvesting opportunities can be completely modified.

5.5 J.L. Clutter. Compatible Growth and Yield Models for Loblolly Pine [8]

We include this more than forty years old article in our review because it introduces the concept of compatible growth and yield models. A model is said compatible when the algebraic form of the yield model can be derived from the growth model. Clutter presents an even-aged, distance independent stand model adjusted to loblolly pine, the most important forest species at that time in the South of U.S.A.. We will study in the following chapter a model based on the one presented here and its extension to a mixed species forest.

Data was obtained from measurements on 102 disease-free loblolly pine stands. They were measured three times, with five years intervals. Approximately half of the stands were thinned at the beginning of the measurement period while the rest were thinned at the time of the second measurement. Each time, the diameters at breast height (d) of all trees were measured and those larger than 1.0 inch were recorded, together with height and age for at least ten sample trees on each stand. The following parameters were computed

- A : Average age during the growth period - computed as the average of the initial and final ages of dominant and co-dominant sampled trees.
- S : Site index, obtained from average age and average height of dominant trees. The site index is used to assess site quality, and it is defined as the top height reached at a specified age such as 20 or 50 years, see [12].
- B : Basal area per acre during the growth period - computed as the average of the initial and final basal area, adding up the area of all trees with diameter of at least 1.0 inch.
- H : Height of every tree was estimated as a function of diameter, fitting the equation $H = a + b_1 d + b_2 d^2 + b_3 d^3$ with data from the sample trees. It is worth mentioning that a separate regression is made for every stand and measurement time, however, the values of

the parameters are not given in the article. The height's estimate of every tree is needed to compute its volume.

V : Average cubic-foot volume during the growth period - computed as the average of the initial and final total cubic-foot volumes per acre (comprising every tree whose diameter is at least 1.0 inch). The inside-bark volume of each tree was obtained from volume tables for loblolly pine, known at the time (Mac-Kinney and Chaiken [20]).

The model

In the first place, the authors develop a regression equation to predict volume per acre as a function of A, S and B. The model chosen is

$$\ln V = b_0 + b_1 S + b_2 \ln B + b_3 A^{-1} \tag{5.7}$$

With the computed values of the parameters ($b_0 = 2.8076$, $b_1 = 0.015108$, $b_2 = 0.95931$ and $b_3 = -21.863$), this equation is showed to account for more that 99% of the variation about mean $\ln V$. Besides, a more general equation including the three two-variables interaction ($\ln V = b_0 + b_1 S + b_2 \ln B + b_3 A^{-1} + b_4 S \ln B + b_5 S A^{-1} + b_6 A^{-1} \ln B$) was fitted to the data. The resulting analysis showed that the interaction variables added, were not significant at the 1%-level. Hence, it is not worth taking them into consideration and the simpler model (5.7) is kept.

To find a dynamic model, the authors derive the growth equation. Direct differentiation of (5.7) with respect to age (remembering that $B = B(A)$ but $S \neq S(A)$) yields the total derivative

$$\frac{dV}{dA} = b_2 V B^{-1} \frac{dB}{dA} - b_3 V A^{-2} \tag{5.8}$$

To deduce the necessary formula for $\frac{dB}{dA}$, the authors resort to the basal area curves presented by Schumacher and Coile [35]. The graphic analysis suggests the following model to relate the basal area growth to functions of site index, age and basal area.

$$\frac{dB}{dA} = B A^{-1} (c_0 + c_1 S - \ln B) \tag{5.9}$$

Equation (5.9) was fitted using multiple regression analysis taking $\frac{dB}{dA} + B A^{-1} \ln B$ as the independent variables and $B A^{-1}$ and $B S A^{-1}$ as the independent ones. Both independent variables are significant at the 1%-level, the regression coefficients are $c_0 = 4.6012$ and $c_1 = 0.013597$ and equation (5.9) accounts for the 65.3% of the variation about the mean of annual basal area growth.

With this specification further development of the volume growth expression is possible: substitution of (5.9) into (5.8) yields immediately

$$\frac{dV}{dA} = -b_3 V A^{-2} - b_2 V A^{-1} \ln B + b_2 c_0 V A^{-1} + b_2 c_1 V S A^{-1}.$$

The authors use (5.7) to estimate the values of V and perform a new regression taking $VA^{-1}\ln B$, VA^{-1}, VSA^{-1} and VA^{-2} as independent variables, finding new values of the coefficients. The fitted equation is

$$\frac{dV}{dA} = d_1VA^{-2} - d_2VA^{-1}\ln B + d_3VA^{-1} + d_4VSA^{-1} \tag{5.10}$$

where $d_1 = 5.7907$, $d_2 = -0.78166$, $d_3 = 3.6562$ and $d_4 = 0.017410$. The last equation accounts for 69.7% of the variation about the mean of the volume growth per acre. Finally, integrating (5.9) and (5.10) we get

$$\begin{aligned} \ln B(A,S) &= c_0 + c_1S - \tfrac{A_0}{A}(c_0 + c_1S - \ln B_0) \\ \ln V(A,S) &= \ln V_0 - \big[d_1 - d_2A_0(c_0 + c_1S\ln B_0)\big]\left(\tfrac{1}{A} - \tfrac{1}{A_0}\right) \\ &\quad + \big[d_3 - d_2c_0 + (d_4 - d_2c_1)S\big]\ln\left(\tfrac{A}{A_0}\right) \end{aligned} \tag{5.11}$$

where B_0 and V_0 stand for $B(A_0, S)$ and $V(A_0, S)$ respectively.

Concluding remarks

Equations (5.11) are the sought equations, useful to compare management alternatives from the point of view of timber volume. The authors compare two possible policies as an example, but they could be used to answer more general question like:

- What is the density at which maximum growth occurs related to age and site?
- With sufficiently high density, does the model allow for the possibility that net basal area or volume growth rates become negative?
- How is the rotation age of maximum mean annual increment related to site, initial density and thinning regime?

It should be noted, though, that long projections of uncut stands may be not realistic, because the data was collected on stands that were thinned at least once during the measurement period. The effect of thinning was not investigated.

5.6 Norihisa Ochi and Quand V. Cao. A comparison of compatible and annual growth models

This article makes a comparison between two “Clutter’ type" compatible models and one annual recursive new model, designed by the authors. We comment it briefly, even though there are no management issues discussed, in order to present more examples of stand growth and yield models.

Although intuitive, the compatibility constraints restrict the number of possible models. The authors relax this constrain and look into a bigger class of models: the so-called numerically compatible models, i.e., models that provide the same growth estimates regardless of length of growth periods. For example, the growth predicted directly from age A_1 to A_3 is identical to the growth predicted in two steps, from A_1 to A_2 and then from A_2 to A_3.

The recursive new model predicts yield based on information from the previous year. The advantage of this new approach is the flexibility allowed in building annual growth models without constraints, while maintaining the step-invariance property.

Data were collected from 162 loblolly pine plots of the South of U.S.A. They comprised the number of surviving trees per hectare (N) and the diameters and heights of 49 trees on each plot that were recorded four times with time intervals varying between 5 and 7 years. Stand height (H), volume (V) and basal area (B) were computed. The data were randomly divided into a fit data set (65%) and a validation data set (35%).

Models

The authors present two published compatible models (Sullivan and Clutter's model and Pienaar and Harrison's model) and propose a new recursive model.

The following height-age equation is proposed for the three models

$$H_{t+q} = \exp\left[\lambda_1 + (\ln H_t - \lambda_1)\left(\frac{A_{t+q}}{A_t}\right)^{\lambda_2}\right]$$

where the parameters are estimated separately from the other equations of the growth and yield systems.

To project the number of surviving trees in the two compatible models, the following equation derived from the Weibull probability density function was selected

$$N_{t+q} = N_t\left(\frac{A_{t+q}}{A_t}\right)^{\alpha_2 - 1} \exp[\alpha_1(A_{t+q}^{\alpha_2} - A_t^{\alpha_2})]$$

The Sullivan and Clutter's model developed for loblolly pine repeats eq. (5.11) of §5.5 for the variation of the basal area (with the values of the parameters fitted to the collected data) and proposes a different equation to estimate the volume

$$\begin{aligned} B_{t+q} &= \exp\left[\frac{A_t}{A_{t+q}} \ln B_t + \left(1 - \frac{A_t}{A_{t+q}}\right)(c_0 + c_1 S)\right] \\ V_{t+q} &= \exp\left[d_1 + d_2 S + \frac{d_3}{A_{t+q}} + d_4 \ln(\hat{B}_{t+q})\right] \end{aligned}$$

The Pienaar and Harrison's model developed for slash pine comprises

$$\begin{aligned}
B_{t+q} &= \exp\big[\ln B_t + \alpha_1\big(\tfrac{1}{A_{t+q}} - \tfrac{1}{A_t}\big) + \alpha_2(\ln \hat{N}_{t+q} - \ln N_t) + \alpha_3(\ln \hat{H}_{t+q} - \ln H_t)\big] \\
&\quad + \alpha_4\big(\tfrac{\ln \hat{N}_{t+q}}{A_{t+q}} - \tfrac{\ln N_t}{A_t}\big) + \alpha_5\big(\tfrac{\ln \hat{H}_{t+q}}{A_{t+q}} - \tfrac{\ln H_t}{A_t}\big) \\
V_{t+q} &= \exp\big[\ln V_t + \beta_1\big(\ln \hat{H}_{t+q} - \ln H_t\big) + \beta_2(\ln \hat{N}_{t+q} - \ln N_t) + \beta_3(\ln \hat{B}_{t+q} - \ln B_t)\big]
\end{aligned}$$

where parameters c_i, d_i, α_i and β_i are fitted with the collected data.

The annual growth model proposed by the authors is defined recursively as

$$\begin{aligned}
N_{t+1} &= \exp\big[\alpha_1 + \ln(N_t)\big(\alpha_2 + \alpha_3\tfrac{A_t}{A_{t+1}}\big) + \ln H_t\big(\alpha_4 + \alpha_5\tfrac{A_t}{A_{t+1}}\big)\big] \\
B_{t+1} &= \exp\big[\tfrac{A_t}{A_{t+1}}\ln B_t + \big(1 - \tfrac{A_t}{A_{t+1}}\big)(\beta_1 + \beta_2 \ln H_t + \beta_3 \ln N_t)\big] \\
V_{t+1} &= \exp\big[\tfrac{A_t}{A_{t+1}}\ln V_t + \big(1 - \tfrac{A_t}{A_{t+1}}\big)(\gamma_1 + \gamma_2 \ln H_t + \gamma_3 \ln N_t + \gamma_2 \ln B_t)\big]
\end{aligned}$$

In the right hand sides, collected data are to be used whenever available.

Concluding remarks

The models were evaluated by means of statistics computed from the validation data set. Evaluations were made for three different projection length. In the article, the values of all the parameters are presented as well as those of the evaluation statistics. The annual model proves to make a better prediction of the forest stand attributes on every test.

It should be noted that compatible growth models are special cases of annual growth models because compatible models can be rewritten in the form of annual growth. In building compatible growth models, it is necessary to restrict oneself to a smaller pool of possible models even at the expense of model predicting ability. Since the class of annual growth models is a superset of the class of compatible models, annual growth models allow more flexibility in selection of independent variables and equation forms

The disadvantage of annual recursive models is that they do not provide a closed expression to estimate the stand attributes as a function of the elapsed time. This could be a difficulty if further calculations are to be made, like for example characterizing the solution to the optimal harvesting problem.

Chapter 6

Clutter-Garcia and Maugé models

In this chapter, we attempt to make the link between two scientific communities that study the forest from different points of view: the forest economics and the forestry. Our objective is to study the optimal harvesting problem when the value function is related with a growth model.

We study two forest growth models, the Clutter-Garcia model and the Maugé model. Both were developed for pine stands: loblolly pine the former and maritime pine the latter, even though it has been adapted for other coniferous trees.

We start by studying their behavior when they evolve freely. Then we study the continuous time Faustmann Problem (compare with (1.1))

$$\max_{T\in\mathbb{R}_+} \sum_{i=1}^{\infty} f(\cdot)e^{-i\delta T} = \frac{f(\cdot)e^{-\delta T}}{1-e^{-\delta T}} \tag{6.1}$$

In Problem (1.1) of Chapter , the function $f(\cdot)$ was given and assumed to be simply a function of the stand age. In this chapter we express it as function of height or mean diameter: $f = f(H(H_0, B_0, T), B(H_0, B_0, T))$, which will be given by the growth model as functions of the elapsed time and the initial conditions. This formulation allows to relate the Faustmann age with the characteristics of the growth process, and then study the effects of a change in the parameters of the model (that could be given by the use of fertilizers, feed additives and pesticides) in the Faustmann age and the maximum benefit.

Finally, we extend the Clutter-Garcia model to two species and study its non-controlled evolution comparing it to the single species case.

6.1 The Clutter-Garcia model

In his work of the nineties Garcia [12] presents a stand level model for a single species even-aged stand. He considers as state variables the top height H and the stand basal area[1] B, forming a two-dimensional state vector (H, B).

He considers for B the equation proposed by Clutter [8] for a loblolly pine stand

$$\frac{dB}{dt} = B(\gamma - \ln B)\frac{1}{t} \tag{6.2}$$

and for the top height increments the following modification of the equation used by Clutter and Lenhart [10] for a site of index 70[2]

$$H(t) = \exp\left(\beta - \frac{1}{\alpha t}\right) \tag{6.3}$$

where $\alpha = 0.0752$, $\beta = 4.78$ and $\gamma = 5.55$. This equation proposed by Garcia approximates the equation of Clutter and Lenhart to within one foot over the range of their data. The height H is measured in foots and B is measured in foots^2/acre. The hypothesis that H is independent of B is often assumed by the foresters as long as the population's density is neither too strong nor too weak.

With the initial condition $B(t_0) = B_0$ and defining the constant $k = \gamma - \ln B_0$ the solution is

$$B(t) = \exp\left(\gamma - \frac{t_0}{t}k\right) = B_0 \exp\left[k\left(1 - \frac{t_0}{t}\right)\right]. \tag{6.4}$$

It is very easy to see that $\lim_{t\to\infty} H(t) = e^{\beta}$ and $\lim_{t\to\infty} B(t) = e^{\gamma}$, for every strictly positive initial condition.

6.1.1 Faustmann's control for the one-species Clutter-Garcia model.

The simplest case

We state the Faustmann problem assuming that the value of the tree stand is proportional to the basal area $f = \theta B$. We assume that at the beginning of each rotation young trees of age t_0 are planted attaining a basal area of B_0. If we denote by T the harvesting period ($T = t - t_0$), then the optimization problem is

$$(P) \quad \max_{T\in I\!R_+} f_1(T) = \frac{\theta B(T+t_0)e^{-\delta T}}{1-e^{-\delta T}}$$

[1] The basal area is the sum of the areas of the stems of all the stand trees measured at human breast height.

[2] The site index is used to assess site quality, and it is defined as the top height reached at a specified age such as 20 or 50 years.

where $B(T+t_0) = B_0 \exp[k(1-\frac{t_0}{T+t_0})]$.

Some properties of $f_1(T)$ we can easily deduce are

- $f_1 \in C^\infty(I\!R_{++}),\ f_1(T) \geq 0.$
- $\lim\limits_{T\to 0^+} f_1(T) = +\infty$ and $\lim\limits_{T\to +\infty} f_1(T) = 0$

The fact that $\lim_{T\to 0} f_1(T) = +\infty$ may suggest that it is optimal to harvest immediately, giving no time to the plants to grow. This is actually an aspect introduced by two issues:

1. the planting cost are not taking into account in our model, hence, it is convenient to sell timber of the young trees we get for free at every time,
2. we assume that the function $f = \theta B$. This is a satisfactory assumption only within a certain domain, a too young forest is certainly out of this valid domain because timber of small trees is worthless.

In the next subsection, we introduce planting cost and different estimations of the function $f(T)$ getting results with a better behavior at $T \approx 0$.

After discarding the infinite value at $T = 0$, we search interior local maxima. We start by finding the critical points of the objective function $f_1(T)$, i.e., we find the zeroes of its derivative, which is equivalent to the following equation

$$(e^{\delta T}-1)B'(T+t_0) = \delta e^{\delta T} B(T+t_0) \iff \tfrac{t_0 k}{\delta} = \tfrac{(T+t_0)^2}{1-e^{-\delta T}}.$$

We define the function $g_1 : I\!R_{++} \longrightarrow I\!R,\ g_1(T) = \frac{(T+t_0)^2}{1-e^{-\delta T}}$ and study its properties to see if it intersects with the constant function equal to $\frac{t_0 k}{\delta}$.

(i) $g_1 \in C^\infty(I\!R_{++}),\ g_1(T) \geq 0$

(ii) $\lim\limits_{T\to 0^+} g_1(T) = +\infty$ and $\lim\limits_{T\to +\infty} g_1(T) = +\infty$

(iii) $g_1'(T) = \frac{T+t_0}{(1-e^{-\delta T})^2}[2(1-e^{-\delta T}) - \delta(T+t_0)e^{-\delta T}]$

(vi) $\lim\limits_{T\to 0^+} g_1'(T) = \lim\limits_{T\to 0^+} \frac{-\delta t_0^2}{(1-e^{\delta T})^2} = -\infty$ and $\lim\limits_{T\to +\infty} g_1'(T) = \lim\limits_{T\to +\infty} 2T = +\infty$

and besides,

$$g_1'(T') = 0 \iff T' + t_0 = \tfrac{2}{\delta}(e^{\delta T'} - 1). \tag{6.5}$$

We observe that due to the convexity of $\frac{2}{\delta}(e^{\delta T'} - 1)$ there is only one positive value of T' satisfying (6.5) and so the function g_1 has a unique critical point which is its global minimum.

It suffices then to check if $g_1(T')$ is less than, equal to or greater than $\frac{t_0 k}{\delta}$ to assure that the objective function $f_1(T)$ has two, one or none critical points respectively. If there are two critical points, it is clear from the properties above that the smaller is a local minimum and the larger is the local maximum we are looking for. We show an example of this case in Figure 6.2. At the bottom we can see function g_1 and its intersection with the constant function $\frac{t_0 k}{\delta}$ at the point where the function f_1 (showed at the top of the figure) has a local maximum. Whenever $g_1(T') \geq \frac{t_0 k}{\delta}$, the function $f_1(T)$ is strictly decreasing.

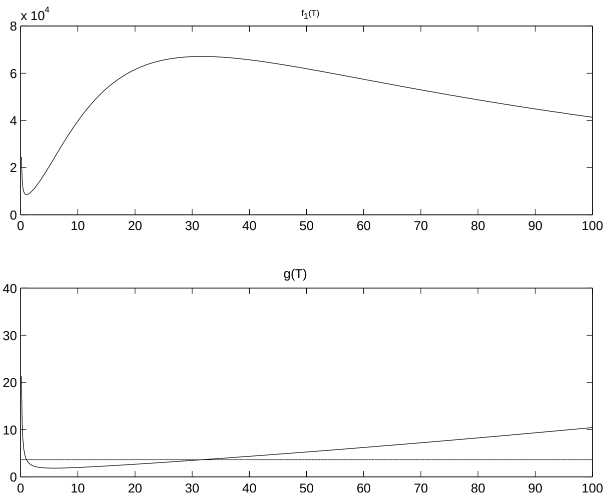

Figure 6.1: Objective function and $g(T)$

We see in (6.5) that the value of T' cannot be found explicitly, and so checking if $g_1(T') < \frac{t_0 k}{\delta}$ is not easy. We try now to find a simpler sufficient condition assuring that there are two critical points. To this aim, some calculation give that $T' < t_0$ and then $g_1(T') < g_1(t_0)$. So, a simple way to assure that $g_1(T') \leq \frac{t_0 k}{\delta}$ is to check if $g_1(t_0) = \frac{4t_0^2}{1-\exp(-\delta t_0)} \leq \frac{t_0 k}{\delta}$.

With the values suggested by Garcia and taking $\delta = \log(1.06)$, i.e., considering a 6% yearly rate, numerical simulation shows that there is no intersection of the curves and f_1 is strictly decreasing.

Introducing planting costs and different stand value functions

There exist in the literature many different equations to estimate the timber volume contained in a stand as a function of stand data as top height and basal area [9, 12]. Frequently, foresters use regression equations of the form

$$V = a + b\,HB \tag{6.6}$$

$$V = a + b\,H^{\alpha_1}B^{\alpha_2} \tag{6.7}$$

where the parameters a, b, α_1 and α_2 are fitted with species and site specific data.

Assuming a constant price p, the stand value function is: $pV(t)$. If we denote ω the cost of planting, the Faustmann problem is stated as

$$(P) \qquad \max_{T\in I\!R_+} \sum_{i=1}^{\infty}[pV(T+t_0)-\omega]e^{-i\delta T} = \frac{e^{-\delta T}}{1-e^{-\delta T}}[pV(T+t_0)-\omega]$$

We assume that $pV(t_0) < \omega < p\lim_{T\to\infty} V(T)$, which means that the planting cost is more than the value of the forest in the market at the planting moment and less than the maximum benefit we can eventually get. Otherwise, the solution is trivial, cut immediately and get infinite profit or let the trees grow to infinity because planting can not be afforded.

Considering equation (6.6) to estimate the volume, the optimization problem is written as

$$(P) \qquad \max_{T\in I\!R_+} f_2(T) = \big\{p[a+bH(T+t_0)B(T+t_0)]-\omega\big\}\frac{e^{-\delta T}}{1-e^{-\delta T}}$$

where the constraints imposed above to ω are expressed as

$$p[a+bH(t_0)B(t_0)] < \omega < p(a+be^{\beta}e^{\gamma}) \tag{6.8}$$

As in the previous subsection, we find the critical points of the function $f_2(T)$

$$\begin{aligned} f_2'(T)=0 &\iff p\,b[H'B+HB'](e^{\delta T}-1) = \big[p\,a+p\,bHB-\omega\big]\delta e^{\delta T} \\ &\iff p\,bHB\big[\tfrac{1}{\alpha(T+t_0)^2}+\tfrac{t_0k}{(T+t_0)^2}\big] = p\,bHB[1-\tfrac{\omega-p\,a}{p\,bHB}]\tfrac{\delta}{1-e^{-\delta T}} \\ &\iff \tfrac{t_0k+1/\alpha}{\delta} = \big[1-\tfrac{\omega-pa}{pb}\exp(-\beta-\gamma+\tfrac{t_0k+1/\alpha}{T+t_0})\big]\tfrac{(T+t_0)^2}{1-e^{-\delta T}} = g_2(T) \qquad (6.9) \end{aligned}$$

Among the properties of $g_2(T)$, we point out

- $g_2 \in C^{\infty}(I\!R_{++})$.
- $\lim_{T\to 0^+} g_2(T) = -\infty$ and $\lim_{T\to+\infty} g_2(T) = +\infty$
- $g_2(T') = 0 \iff T' = \frac{t_0k+1/\alpha}{\gamma+\beta+\log(pb)-\log(\omega-pa)} - t_0$

where $T' > 0$, as it follows from (6.8). Evidently, (6.9) has at least one solution and it is greater than T'. A cumbersome computations shows

$$g_2'(T) = g_2(T)\big[\tfrac{2}{T+t_0}-\tfrac{\delta}{e^{\delta T}-1}\big] + \tfrac{t_0k+1/\alpha}{1-e^{-\delta T}}\,\tfrac{\omega-pa}{pb}\exp(-\beta-\gamma+\tfrac{t_0k+1/\alpha}{T+t_0})$$

We see in Figure 6.2 an example of functions $g_2(T)$ and $g_2'(T)$. Some of the properties of $g_2'(T)$ are:

- $\lim_{T\to 0^+} g_2'(T) = +\infty$ and $\lim_{T\to +\infty} g_2'(T) = +\infty$

Proving that $g_2'(T) > 0$ for all $t > T'$, assures that (6.9) has a unique solution. Although numerical simulations suggest that this is true, it is not easy to prove for arbitrary values of the parameters.

We try a different method: (6.9) is equivalent to

$$\frac{t_0 k + 1/\alpha}{\delta}(1 - e^{-\delta T}) = (T + t_0)^2 \Big[1 - \frac{\omega - pa}{pb} \exp\big(-\beta - \gamma + \frac{t_0 k + 1/\alpha}{T + t_0}\big)\Big] \tag{6.10}$$

We define the functions

$$\begin{aligned} h_1(T) &= \frac{t_0 k + 1/\alpha}{\delta}(1 - e^{-\delta T}) \quad \text{and} \\ h_2(T) &= (T + t_0)^2 \Big[1 - \frac{\omega - pa}{pb} \exp\big(-\beta - \gamma + \frac{t_0 k 1 + /\alpha}{T + t_0}\big)\Big]. \end{aligned}$$

It is easy to see that $h_1(T)$ is strictly positive, strictly increasing and strictly concave:

- $h_1'(T) = (t_0 k + 1/\alpha)e^{-\delta T} > 0 \iff h_1(T)$ is strictly increasing.
- $h_1''(T) = -\delta(t_0 k + 1/\alpha)e^{-\delta T} < 0 \iff h_1(T)$ is strictly concave.

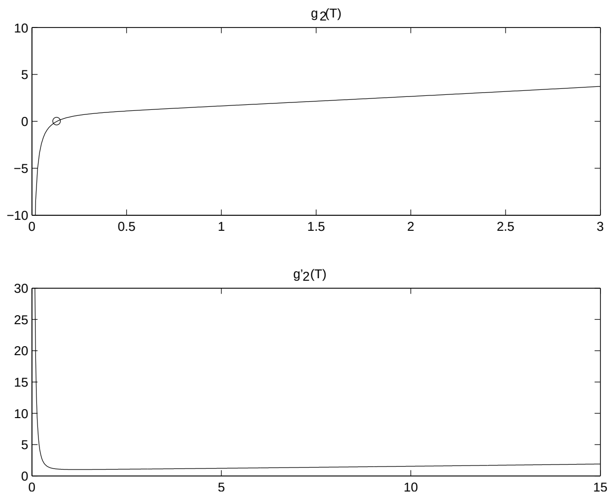

Figure 6.2: $g_2(T)$ and $g_2(T')$.

Obviously $h_2(T)$ has a unique zero at T'. As the expression in brackets is positive and strictly increasing with respect to T for $T > T'$, it follows that $h_2(T)$ is also positive and strictly increasing in this interval. We aim to show now that $h_2(T)$ is strictly convex, in order to deduce that there is a unique point in $I\!R_+$ satisfying (6.10), which must belong to (T', ∞).

The first derivative is

$$h_2'(T) = 2(T+t_0)\big[1 - \tfrac{\omega-pa}{pb}\exp\left(-\beta-\gamma+\tfrac{t_0k+1/\alpha}{T+t_0}\right)\big] + (t_0k+1/\alpha)\tfrac{\omega-pa}{pb}\exp\left(-\beta-\gamma+\tfrac{t_0k+1/\alpha}{T+t_0}\right)$$

and the second derivative is

$$h_2''(T) = 2 - \tfrac{\omega-pa}{pb}\exp\left(-\beta-\gamma\right)\exp\left(\tfrac{t_0k+1/\alpha}{T+t_0}\right)\left[2-2\left(\tfrac{t_0k+1/\alpha}{T+t_0}\right) - \left(\tfrac{t_0k+1/\alpha}{T+t_0}\right)^2\right].$$

We proceed to check that $h_2''(T) > 0$ for $T > T'$. It follows directly if the expression between brackets is negative. If this is not the case, we have

$$\tfrac{\omega-pa}{pb}\exp\left(-\beta-\gamma\right) = \exp\left(-\tfrac{t_0k+1/\alpha}{T'+t_0}\right) < \exp\left(-\tfrac{t_0k+1/\alpha}{T+t_0}\right) \quad \text{for all } T>T'$$

where the equality follows from the definition of T'. Using that the expression in brackets is non-negative, we have

$$h_2''(T) \geq 2 - \left[2-2\left(\tfrac{t_0k+1/\alpha}{T+t_0}\right) - \left(\tfrac{t_0k+1/\alpha}{T+t_0}\right)^2\right] = 2\left(\tfrac{t_0k+1/\alpha}{T+t_0}\right) + \left(\tfrac{t_0k+1/\alpha}{T+t_0}\right)^2 > 0$$

We proved that (6.9) has a unique solution and that it belongs to (T', ∞). The following properties of f_2 imply that such solution of (6.9) is the unique maximum of f_2.

- $f_2 \in C^\infty(I\!R_{++})$.
- $\lim_{T\to 0^+} f_2(T) = -\infty$ and $\lim_{T\to +\infty} f_2(T) = 0^+$

Other volume estimations

If the volume is estimated through the more general equation (6.7), the previous study can be easily extended. Indeed, the new problem can be stated as

$$(\tilde{P}) \quad \max_{T\in I\!R_+} \tilde{f}_2(T) = \left[p(a + b\,H^{\alpha_1}(T+t_0)B^{\alpha_2}(T+t_0)) - \omega\right]\tfrac{e^{-\delta T}}{1-e^{-\delta T}}$$

where $p(a + bH_0^{\alpha_1}B_0^{\alpha_2}) < \omega < p(a + b\exp(\alpha_1\beta+\alpha_2\gamma))$. Following the steps of the previous section, we can see that the critical points of the objective function are now characterized by

$$\tfrac{\alpha_2 t_0 k+\alpha_1/\alpha}{\delta} = \tfrac{(T+t_0)^2}{1-e^{-\delta T}}\big[1 - \tfrac{\omega-pa}{pb}\exp\left(-\alpha_1\beta - \alpha_2\gamma + \tfrac{\alpha_2 t_0 k+\alpha_1/\alpha}{T+t_0}\right)\big],$$

and that the functions

$$\tilde{h}_1(T)=\tfrac{\alpha_2 t_0 k+\alpha_1/\alpha}{\delta}(1-e^{-\delta T}),$$
$$\tilde{h}_2(T)=(T + t_0)^2\big[1-\tfrac{\omega-pa}{pb}\exp\left(-\alpha_1\beta - \alpha_2\gamma + \tfrac{\alpha_2 t_0 k+\alpha_1/\alpha}{T+t_0}\right)\big]$$

have a unique intersection point to the right of $T' = \frac{\alpha_2 t_0 k+\alpha_1/\alpha}{\alpha_2\gamma+\log(pb)-\log(\omega-pa)+\alpha_1\beta/\alpha} - t_0 > 0$

Particular Case *w = f(0)*

We study a limiting case that behaves differently than the general problem. Assuming that the value of a stand is proportional to $B(T)$ and that the planting cost is equal to the stand value at $T = 0$, the Faustmann problem is stated as

$$(P) \quad \max_{T \in I\!R_+} f_3(T) = \frac{\theta[B(T+t_0)-B_0]e^{-\delta T}}{1-e^{-\delta T}}.$$

Some of the properties of $f_3(T)$ are

- $f_3 \in C^\infty(I\!R_{++})$ and $f_3(T) \geq 0$
- $\lim_{T \to 0^+} f_3(T) = \frac{\theta e^\gamma}{\delta t_0}$ and $\lim_{T \to +\infty} f_3(T) = 0$

To solve this one dimensional optimization problem, we find the critical points of the objective function, which are characterized by the following equation

$$(e^{\delta T}-1)B'(T+t_0) = \delta e^{\delta T}[B(T+t_0) - B(t_0)] \iff \frac{t_0 k}{\delta} = \frac{(T+t_0)^2}{1-e^{-\delta T}}\big[1 - \exp(\tfrac{-Tk}{T+t_0})\big].$$

As before, we define the function $g_3(T) = \big[1 - \exp\frac{-Tk}{T+t_0}\big]\frac{(T+t_0)^2}{1-e^{-\delta T}}$. We see in Figure 6.3 the graph of an example of $g_3(T)$ and $g_3'(T)$ and we give a list of some properties.

- $\lim_{T \to 0} g_3(T) = \frac{t_0 k}{\delta}$ and $\lim_{T \to +\infty} g_3(T) = +\infty$
- $g_3'(T) = g_3(T)[\frac{2}{T+t_0} - \frac{\delta}{e^{\delta T}-1}] + \frac{t_0 k}{1-e^{-\delta T}} \exp\big(\frac{-Tk}{T+t_0}\big)$
- $\lim_{T \to 0} g_3'(T) = \frac{k}{\delta}(t_0\delta - k + 2)$ and $\lim_{T \to +\infty} g_3'(T) = +\infty$

To assure that there is at least one value of T different from 0 such that $g_3(T) = \frac{t_0 k}{\delta}$, it suffices to require $\lim_{T \to 0} g_3'(T) < 0$ which is equivalent to $(t_0\delta + 2) < k$.

The uniqueness of such a T is a bit more delicate, repeating the process of the previous section

$$\frac{t_0 k}{\delta} = g_3(T) \iff \frac{t_0 k}{\delta}(1 - e^{-\delta T}) = (T + t_0)^2\big[1 - \exp\big(\tfrac{-kT}{T+t_0}\big)\big].$$

Defining $h_1(T)$ as before and $h_3(T) = (T + t_0)^2\big[1 - \exp\big(\frac{-kT}{T+t_0}\big)\big]$, we can see that $h_3(T)$ is increasing and convex for all $T > 0$ and as before the uniqueness of the intersection point is assured.

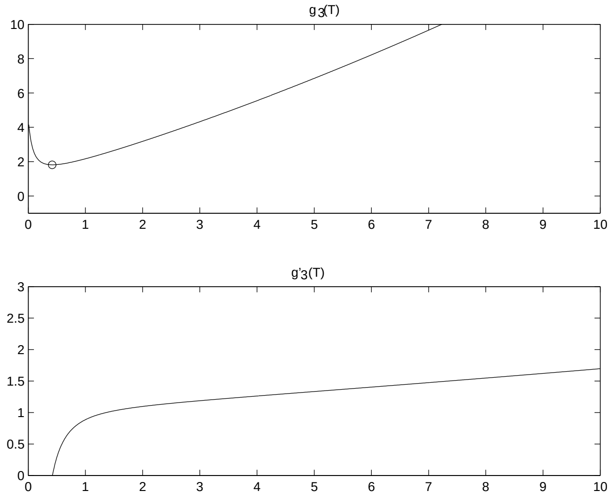

Figure 6.3: $g_3(T)$ and $g_3'(T)$ when $w = f(0) \quad (k = 10)$.

6.1.2 Two species

We propose an extension to two species of the Clutter-Garcia model, where the two species are related through the space competition. This space competition is reflected in the basal area but not in top height which would be affected by light competition.

$$
\begin{aligned}
H_1(t) &= \exp\left(\beta_1 - \tfrac{1}{\alpha_1 t}\right) \\
H_2(t) &= \exp\left(\beta_2 - \tfrac{1}{\alpha_2 t}\right) \\
\tfrac{dB_1}{dt} &= B_1\left[\gamma_1 - \ln(B_1+B_2)\right]\tfrac{1}{t} \\
\tfrac{dB_2}{dt} &= B_2\left[\gamma_2 - \ln(B_1+B_2)\right]\tfrac{1}{t}
\end{aligned}
\tag{6.11}
$$

with initial conditions $B_1(t_0) = B_1^0$ and $B_2(t_0) = B_2^0$. The term $(\gamma_i - \ln(B_1 + B_2))$ aims to represent the space competition, meaning that a slower growth rate is expected at higher stand density. When the total biomass is small, growth is linear because the term $B_i\gamma_i$ dominates. When the biomass becomes larger, space competition arises and growth becomes slower.

Once this term is introduced, we cannot find a closed form for the functions $B_i(t),\ i=1,2$, except in the special case $\gamma_1=\gamma_2=\gamma$. In the general case, we can see that when $\gamma_1>\gamma_2$ species 1 completely suffocates species 2 for every initial condition.

Special case: $\gamma_1 = \gamma_2 = \gamma$

This special case models either a forest composed by two groups of trees of the same species, i.e., $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$, but with different initial conditions, or a mixture of two different species with the particularity $\gamma_1 = \gamma_2$.

Adding the differential equations corresponding to B_1 and B_2 we get

$$\frac{dB_1+B_2}{dt} = \left(\frac{B_1+B_2}{t}\right)[\gamma - \ln(B_1+B_2)]$$

and we solve it easily to find

$$(B_1+B_2)(t) = \exp\left[\gamma - \tfrac{t_0}{t}(\gamma - \ln(B_1^0+B_2^0))\right]$$

and finally

$$\begin{aligned} B_1(t) &= B_1^0 \exp\left[(\gamma - \ln(B_1^0+B_2^0))\left(1 - \tfrac{t_0}{t}\right)\right] \\ B_2(t) &= B_2^0 \exp\left[(\gamma - \ln(B_1^0+B_2^0))\left(1 - \tfrac{t_0}{t}\right)\right] \end{aligned}$$

so that the proportion of area occupied by each species is constant with respect to time, $\frac{B_1(t)}{B_2(t)} = \frac{B_1^0}{B_2^0}$.

Asymptotic behavior in the general case

We claim that when $\gamma_1 > \gamma_2$, the species 1 completely suffocates the species 2, whose basal area converges to 0.

We can state the system of ordinary differential equations given in (6.11) as an autonomous system by changing the independent variable $u = \log(t)$ (notice that $\lim_{t\to\infty} u(t) = +\infty$ hence the asymptotic limits remain unchanged). The new system is simply:

$$\text{(6.12)} \qquad \left\{ \begin{aligned} \frac{dB_1}{du} &= B_1\left[\gamma_1 - \ln(B_1+B_2)\right] \\ \frac{dB_2}{du} &= B_2\left[\gamma_2 - \ln(B_1+B_2)\right] \end{aligned} \right.$$

Figure 6.4 shows the phase diagram of (6.12), and it is easy to see that $\lim_{t\to\infty} B_1(t) = e^{\gamma_1}$ and $\lim_{t\to\infty} B_2(t) = 0$.

It may be argued that having $H_2 \to e^{\beta_2}$ and $B_2 \to 0$ is as a contradiction. It is explained because we have not included any term of competition for the state variables H_i. On the contrary, H_i behaves independently of all the others state variables (in fact, it evolves as in the one species case) and is strictly increasing. But, as H_i is defined as the top height of the species i (the height of the tallest tree), there is not continuity required in the limit of this variable and our model still makes sense.

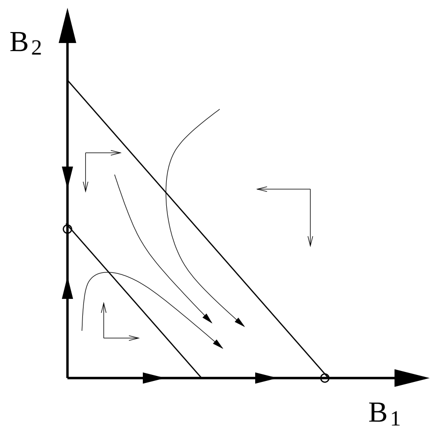

Figure 6.4: Phase Diagram B_1 vs. B_2.

6.2 Maugé's model

This is a discrete time stand model that was initially calibrated for maritime pine and has been adapted afterwards for other types of coniferous trees. The variables are height (H), circumference (C) and number of trees (N), although in its simplest form N remains constant.

Variables:				Parameters:			
	C	–	circumference		$\bar{H}$	–	maximal height
	H	–	height		a	–	seasonal fertility
	N	–	number of trees		b	–	water availability
					k	–	water availability (k is such that $kNC \leq 1$).

Equations:

$$\begin{cases} \Delta H &= a(\bar{H} - H) \\ \Delta C &= b\Delta H(1 - kNC) \end{cases}$$

The term $b\Delta H$ represents the diameter increase without competition, while $(1 - kNC)$ aims at modelling the intra-species space-competition effect. We study its continuous time version:

$$\begin{cases} \frac{dH(x)}{dx} &= a[\bar{H} - H(x)] \\ \frac{dC(x)}{dx} &= ab[\bar{H} - H(x)][1 - kNC(x)]. \end{cases} \tag{6.13}$$

As the number of stems is considered constant we can merge N with the constant k, i.e. from now on we denote the product kN simply by k. We can eliminate the constant a from the calculations by making the change of variable $t = ax$, getting the somewhat simpler expression

$$\begin{cases} H'(t) &= \bar{H} - H(t) \\ C'(t) &= b[\bar{H} - H(t)][1 - kC(t)] \end{cases}$$

As H is generally considered independent from the silviculture we will not take the initial height as a possible decision variable. We take simply $H(0)=0$ to get $H(t)=\bar{H}(1-e^{-t})$ and thus the differential equation that characterizes $C(t)$ is

$$C'(t) = b\,e^{-t}\,(1-kC) \tag{6.14}$$

Denoting $C(t_0) = C_0$, we get

$$C(t) = \tfrac{1}{k}\big\{1-(1-kC_0)\exp[\,kb\,(e^{-t}-e^{-t_0})]\big\} = \tfrac{1}{k}\big\{1-\alpha\exp(\,kb\,e^{-t})\big\} \tag{6.15}$$

with $\alpha = (1-kC_0)\exp(-\,kb\,e^{-t_0}) > 0$. It is easy to see that the limits are:

$$\begin{aligned} \lim_{t\to\infty} H(t) &= \bar{H} \\ \lim_{t\to\infty} C(t) &= \tfrac{1-\alpha}{k} \end{aligned}$$

thus, there is not a global attractor as the limit of $C(t)$ when $t \to \infty$ depends on the value of the parameters and the initial conditions. Notice that it is not $\lim_{t\to\infty} C(t) = \frac{1}{k}$ as one could expect by making the rhs equal to 0, as would be the process if (6.14) was autonomous. Observe that $\lim_{t\to\infty} C(t) < \frac{1}{k}$ except if $C_0 = \frac{1}{k}$ which yields the trivial solution $C(t) \equiv \frac{1}{k}$

6.2.1 Faustmann's control for the one-species Maugé model

We state the Faustmann Problem assuming that the value of the tree stand is proportional to the mean circumference of the trees $f = \theta C$. We assume that at the beginning of each rotation young trees of age t_0 are planted with an initial circumference C_0, and the optimization problem can be stated as

$$(P) \quad \max_{T\in I\!R_+} f_1(T) = \frac{\theta C(T+t_0)e^{-\delta T}}{1-e^{-\delta T}} = \frac{\theta C(T+t_0)}{e^{\delta T}-1}$$

where $C(T+t_0) = \frac{1}{k}\big[1-\alpha\exp(\,kb\,e^{-(T+t_0)})\big]$. Some of the properties of $f_1(T)$ are:

- $f_1 \in C^\infty(I\!R_{++}),\ \ f_1(T) \geq 0.$
- $\lim_{T\to 0^+} f_1(T) = +\infty$ and $\lim_{T\to+\infty} f_1(T) = 0$

We look for the interior critical points of the objective function $f_1(T)$. The calculations are somewhat more involved than in the previous section.

$$\begin{aligned} f_1'(T) = 0 \quad &\Longleftrightarrow \quad (e^{\delta T}-1)C'(T+t_0) = \delta e^{\delta T} C(T+t_0) \\ &\Longleftrightarrow \quad b\,e^{-(T+t_0)}\big(1-kC(T+t_0)\big) = \tfrac{\delta}{1-e^{-\delta T}} C(T+t_0) \\ &\Longleftrightarrow \quad \tfrac{\delta}{k} = (1-kC(T+t_0))\big[be^{-(T+t_0)}(1-e^{-\delta T}) + \tfrac{\delta}{k}\big] \qquad (6.16) \end{aligned}$$

We define the function $g_1(T)$ equal to the rhs of the previous equation, and we study if it intersects the constant function equal to $\frac{\delta}{k}$.

- $g_1(T)$ is continuous.
- $g_1(0) = \delta(\frac{1}{k} - C_0)$
- $\lim_{T\to\infty} g_1(T) = \delta(\frac{1}{k} - C_0)\exp(-kbe^{-t_0})$

Observe that $\frac{\delta}{k} > g_1(0)$. In fact, there is no intersection point, because $g_1(T)$ is strictly decreasing as we can see by proving that $g_1'(T) < 0$

$$\begin{aligned} g_1'(T) &= -C'(\cdot)\big[kbe^{-(T+t_0)}(1-e^{-\delta T}) + \delta\big] + \big(1 - kC(\cdot)\big)be^{-(T+t_0)}\big[-(1-e^{-\delta T}) + \delta e^{-\delta T}\big] \\ &= -be^{-(T+t_0)}\big(1 - kC(\cdot)\big)\big[kbe^{-(T+t_0)}(1-e^{-\delta T}) + \delta + (1-e^{-\delta T}) - \delta e^{-\delta T}\big] \\ &= -be^{-(T+t_0)}\big(1 - kC(\cdot)\big)(1-e^{-\delta T})\big[kbe^{-(T+t_0)} + 1 + \delta\big] < 0 \end{aligned}$$

which is obviously strictly negative. From this, it follows that $f_1(T)$ has no interior critical points and is strictly decreasing. The study of the model suggests that the optimal policy consists in harvesting at $T = 0$. As in the previous section, this is a solution introduced because the planting cost is neglected and because of the assumption about the form of the value function. In the following we introduce planting cost and see that we can retrieve a solution.

Introducing planting cost

We state the Faustmann Problem assuming that the timber volume contained in the forest is proportional to $C(t)$, the price p is constant and ω is the planting cost. We consider that young trees are planted at age t_0 with a circumference C_0.

The planting cost should be greater than the minimum benefit that can be obtained at each harvesting time and less than the corresponding maximum

$$p\theta C_0 < \omega < \tfrac{p\theta}{k}[1 - (1 - kC_0)\exp(kb\, e^{-t_0})] \tag{6.17}$$

Otherwise, the solution is trivial, cut immediately and get infinite profit in the first case or let the trees grow to infinity because planting can not be afforded.

We will try to determine the optimal period, this is:

$$(P) \quad \max_{T\in I\!R_+} f_2(T) = \max_{T\in I\!R_+} \tfrac{e^{-\delta T}}{1-e^{-\delta T}}\big[p\theta\, C(T+t_0) - \omega\big].$$

And characterizing the critical points of the objective function is equivalent to

$$\tfrac{p\theta C'(T+t_0)}{p\theta C(T+t_0)-\omega} = \tfrac{\delta}{1-\exp(-\delta T)} \tag{6.18}$$

after some calculation we get

$$(6.18) \iff \delta\big(\tfrac{1}{k} - \tfrac{\omega}{p\theta}\big) = (1 - kC(T+t_0))\big[be^{-(T+t_0)}(1 - e^{-\delta T}) + \tfrac{\delta}{k}\big] = g_1(T).$$

Notice that by making $\omega = 0$ we retrieve (6.16). The continuity of g_1 together with (6.17), imply that there is always at least one solution to eq. (6.18), since

$$p\theta C_0 \leq \omega \leq \frac{p\theta}{k}[1 - (1 - kC_0)\exp(kb\, e^{-t_0})] \iff g(0) \geq \delta\left(\frac{1}{k} - \frac{\omega}{p\theta}\right) \geq \lim_{T\to\infty} g(T)$$

and such solution is unique, because $g_1(T)$ is strictly decreasing.

The study with different volume estimation functions becomes much more involved than in the previous section and has not yet give useful results.

The model is not suitable to be extended to two species with space competition, because its variables are height and mean circumference of each species. To relate the mean circumference with the space available we would need to know the proportion of each species within the population and its variation. However, this is not possible as the evolution of the number of stems per area has been neglected.

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